Chow schemes in mixed characteristic

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by
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Abstract

In this thesis we compare Suslin–Voevodsky's sheaves of proper effective relative cycles with presheaves representable by certain monoid objects. We give two results in this direction; the first describes a higher dimensional analogue of Suslin–Voevodsky's comparison between relative zero cycles and the graded monoid of symmetric powers ([SV96, Thm. 6.8]) and the second is a new proof of a direct generalization of loc.cit.

The key component of our efforts is a theorem, proved on the way, telling us that after restricting ourselves to seminormal schemes the morphism from the presheaf represented by a commutative-monoid object (satisfying reasonable assumptions) to its sheafification in the h-topology, becomes an isomorphism after appropriate extension of scalars. Furthermore we also introduce a construction, developed in collaboration with Jarle Stavnes, which allows for a uniform (and sometimes simplified) study of the theory of semi and (absolute) weak normality. We then apply this construction to obtain several interesting results concerning the twin theories of weak and semi-normality in a parallel manner. Moreover our construction together with its consequences allow us to describe representable sheaves in a large family of h-topologies, without having to do a case by case study. In particular this reproves a special case of Rydh's description of representable sheaves in the h-topology, as well as extends Huber–Kelly's analogous result for the decomposable h-topologies.

This thesis was written with the additional purpose of providing a self contained presentation of the theory of relative cycles and the construction of the Chow scheme. To achieve this we recall many of the definitions and results from [SV00] and occasionally expand on the explanations found there.

Contents

Introduction

Historical background

Spaces parametrizing positive algebraic cycles have been in use in algebraic geometry for a long while. For instance Severi, who initially introduced the notion of rational equivalence of cycles, discovered that positive zero cycles on a surface correspond to points on the symmetric powers of this surface. Severi then used this parametrization to describe rational equivalence of cycles by means of rational curves on the symmetric powers.

Later Chow proved the existence of a variety, today called the Chow variety, which parametrizes cycles of dimension r and degree d on a given variety. Just as in the case of zero cycles, rational curves on the Chow variety describe rational equivalence of cycles (see [Sam56, Thm. 3]).

Many years after their introduction, Chow varieties were used to define Lawson homology (see [Law89], [Fri91]) and were also applied in the study of rationally connected varieties (see [Kol96, Ch. IV, Sec. 3]).

Moreover as part of a quest to construct a reasonable singular homology theory on the category of schemes of finite type over an arbitrary field k, Suslin–Voevodsky introduced the concept of relative zero cycles in [SV96]. Although speaking of a cycle on a variety parametrized by the points on another variety is nothing new (see for instance [58]), the fact that one can define a presheaf of such cycles (with the correct extra constraints) on the category of normal varieties was certainly novel. Furthermore they proved that the presheaf of effective relative cycles on X is isomorphic to the presheaf represented by infinite symmetric powers of X (after localization by the characteristic of the ground field, when it is positive). This gives an interpretation of the symmetric powers parametrizing positive zero cycles which is more in line with Grothendieck's idea that parameter spaces in algebraic geometry should ideally be solutions to moduli problems.

Suslin–Voevodsky later extended their theory of relative cycles to higher dimensional cycles and also to all Noetherian schemes in [SV00]. Their paper also gives a modern treatment of the Chow scheme as a scheme representing relative effective cycles in the h-topology, a rather fine topology previously introduced by Voevodsky. The theory of sheaves of relative cycles is one of the main technical tools which Voevodsky used to construct a triangulated theory of motives over a field thus opening a vibrant area of research. For more on the story of mixed motives see [VSF00].

Central concepts

Relative cycles, the h-topology and seminormality are arguably the three most central notions of this thesis. Let us now give a flavour of each of these concepts.

Relative cycles

For a scheme $X \to S$ of finite type over a Noetherian scheme S a relative cycle is informally speaking a cycle (i.e. a formal sum of points) on X which lies over generic points of S and has a well defined specialization to any fiber of $X \to S$. These cycles are especially of interest because they allow for well defined base change to any Noetherian scheme S' over S. This fact is due to deep connections between the theory of properness and flatness which is then combined with Galois theory. We will explain this in more detail in the course of this thesis.

The h-topology

The h-topology is a Grothendieck topology on the category of schemes which is finer than most of the topologies appearing in algebraic geometry. As one might expect this property has its pros and cons. A good property is that if one has a sheaf in the h-topology then we have very many ways of gluing sections together which can make the search for a section with certain properties easier. On the other hand it is a rather strong condition to be an h-sheaf and there are several functors represented by rather reasonable schemes which fail to be sheaves in the h-topology.

Seminormality

The existence of a variety giving the "best approximation to the normalization" of a given complex variety under the constraint of keeping the same topological space was first proven by Andreotti-Norguet in [AN67]. They called this construction *weak normalization*. Later Andreotti-Bombieri developed the weak normalization in the context of schemes in [AB69] where they proved that for a dominant morphism $f : Y \to X$ there exists a universal homeomorphism $\sigma : {}^*X \to X$ initial among universal homeomorphisms with target X that f factors through. This map σ is the *weak normalization* of X in Y. Similarly Traverso proved in [Tra70] that one also has a factorization $Y \to {}^+X \xrightarrow{\sigma'} X$ where the last morphism σ' is a universal homeomorphism such that the induced maps of residue fields are isomorphisms and σ' is initial among such morphisms that f factors through. In this case σ' is said to be the *seminormalization* of

X in Y. We let $X^{sn} \to X$ (resp. $X^{wn} \to X$) denote the seminormalization (resp. weak normalization) of X in its integral closure and call this the *seminormalization* (resp. *weak normalization*) of X. We say that X is *seminormal* (resp. *weakly normal*) if X is isomorphic to its seminormalization (resp. weak normalization). In characteristic zero these twin notions coincide, but not in general. Furthermore seminormalization is a functorial operation, while weak normalization is not (a counter example can be found in the proof of [Kol96, Ch. 1, Prop.7.2.3]).

In [Ryd10] the notion of *absolute weak normality* is introduced where the absolute weak normalization of a scheme X is a universal homeomorphism $\sigma : X^{awn} \to X$ such that if $\sigma' : Y \to X$ is any other universal homeomorphism then σ factors through σ'. The absolute weak normalization is functorial.

Main results

The central purpose of this thesis is to prove Theorem 6.5.3 and Theorem 7.2.1. We now state the first of these two main theorems:

Theorem (6.5.3). *Let S be a Noetherian scheme and $i : X \to \mathbb{P}^n_S$ a closed embedding. Then there exists a monoid object in the category of schemes over S, denoted by $C_r((X,i)/S)$, such that if* $\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+}$ *denotes Suslin-Voevodsky's presheaf of proper effective relative cycles with coefficients in $\mathbb{Q}_+$[1] then after restricting this presheaf and $h_{C_r((X,i)/S)}$ to the category of semi-normal Noetherian schemes over S we have an isomorphism of presheaves of monoids:*

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+} \to h_{C_r((X,i)/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+.$$

For a morphism of schemes $X \to S$ and positive integer d we denote the d'th symmetric power (when it exists) by $\mathrm{Sym}^d(X/S)$. The following theorem is our second main result of this thesis.

Theorem (7.2.1). *Let $X \to S$ be a flat finite type morphism to a Noetherian scheme S such that X/S is* AF *(Definition 1.5.26) and let* $\mathrm{Sym}^\bullet(X/S)$ *denote the commutative monoid* $\coprod_{d\geq 0} \mathrm{Sym}^d(X/S)$. *Then after restricting the presheaves of monoids* $h_{\mathrm{Sym}^\bullet(X/S)}$ *and* $\mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{\mathbb{Q}_+}$ *to the category of Noetherian semi-normal schemes over S we get an isomorphism of presheaves of monoids:*

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{\mathbb{Q}_+} \to h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+. \tag{0.0.1}$$

We emphasize that combining [Ryd08b, Thm.3.1.11], [Ryd08b, Cor.4.2.5] and [Ryd08a, Thm. 10.17] one obtains another proof of Theorem 7.2.1. Furthermore this theorem is a generalization of Theorem 6.8 of [SV96], which is to

[1] This is the sub-monoid of $\mathbb{Q}$ consisting of the non-negative rational numbers.

the best of our knowledge the first modern (functorial) comparison between effective zero cycles and symmetric powers. The set-up of loc.cit. is as follows: The scheme S is any field k and the presheaves involved are restricted to the category of normal varieties over k. The statement of their Theorem is also a little more specific in the sense that they describe the isomorphism by means of "symmetrization", a construction that involves providing a section to the projection from symmetric powers of a scheme finite and surjective over a normal scheme. The use of this symmetrization construction is taken further in Chapter 3 of [Har16], not only by considering symmetrization in the relative set-up, but by also showing that it is compatible with a lot of additional structure on both sides which is not considered in [SV96]. Proposition 7.2.4 tells us that our isomorphism from Theorem 7.2.1 coincides with symmetrization after restricting ourselves to normal Noetherian schemes.

Strategy

Our two main theorems are proved in a similar manner: first we sheafify our presheaves in a fine topology such as the h-topology. This gives a lot of freedom to glue sections together which makes it possible to construct homomorphisms between our sheaves which in fact turn out to be isomorphisms. The final step then involves understanding how our presheaves compare to their sheafifications. This turns out to be described by means of purely inseparable field extensions which is why we want the seminormality assumption as one has more control over field arithmetic in that setting.

The Chow monoid

For a closed embedding $i : X \to \mathbb{P}^n_S$ the presheaf of relative cycles of degree d and dimension r is representable by a scheme $C_{r,d}((X,i)/S)$ in the h-topology. The infinite coproduct of these schemes (as d varies) can be endowed with the structure of a commutative monoid object in the category of schemes which we denote by $C_r((X,i)/S)$ and call the *Chow monoid* of r cycles with respect to i. This monoid scheme h-represents the presheaf of relative r cycles (Theorem 6.4.2).

Representable sheaves in the h-topologies

The following theorem gives a neat description of the sheafification of a representable functor in the h-topology.

Theorem (Rydh)**.** *Let X be an algebraic space locally of finite presentation over a scheme S and let T be a quasi-compact and quasi-separated scheme over S. Then if $\mathrm{L}_h(X/S)$ denotes the sheafification of $h_{X/S}$ with respect to the h-topology we have*

$$\mathrm{L}_h(X/S)(T) = \mathrm{Hom}_S(T^{awn}, X) = \mathrm{colim}_\lambda \mathrm{Hom}_S(T_\lambda, X),$$

where T^{awn} denotes the absolute weak normalization of T and the colimit is taken over all finitely presented universal homeomorphisms $T_\lambda \to T$.

If one instead works with one of the coarser cousins of the h-topology where one has more control over field arithmetic, for instance the *cdh*-topology introduced in [SV00], one can then understand the sheafification of a representable presheaf in terms of the seminormalization even in mixed characteristic as long as the base scheme is reasonably nice. This is the content of [HK18, Proposition 6.14] which we now recall precisely:

Theorem (Huber–Kelly). *Suppose the Noetherian base scheme S is Nagata. Then for any two schemes X, T of finite type over S we have $\mathrm{L}_{cdh}(X/S)(T) = \mathrm{Hom}_S(T^{sn}, X)$. Furthermore the natural maps*

$$\mathrm{L}_{rh}(X/S) \to \mathrm{L}_{cdh}(X/S) \to \mathrm{L}_{eh}(X/S) \to \mathrm{L}_{sdh}(X/S) \to h_{X/S_{val}}$$

are isomorphisms of presheaves on the category of finite type S-schemes.

The similarity of these two theorems fits in a paradigm where semi and (absolute) weak normality frequently possess similar properties. This observation led to a construction, developed in collaboration with Stavnes, whose purpose is to yield a uniform theory of the twin notions of semi and (absolute) weak normality. One of the applications of our theory is Theorem 4.3.9 which both essentially extends [HK18, Proposition 6.14] to all Noetherian schemes over S and also gives a special case of Rydh's description of representable sheaves in the h topology.

We previously mentioned that a vital step in the proofs of our main theorems is to understand how certain representable functors compare to their sheafifications. This is achieved in the following theorem:

Theorem (5.0.1). *Let S be a Noetherian scheme and M/S be a commutative monoid object in the category of schemes over S and t be any Grothendieck topology finer than the uh topology and coarser than the h topology. Suppose further that the morphism $M \to S$ is flat, locally of finite type and* AF *(Definition 1.5.26). Then after restricting the presheaves $h_{M/S}$ and its t-sheafification $\mathrm{L}_t(M/S)$ to the category of seminormal Noetherian S-schemes the natural map*

$$h_{M/S} \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{L}_t(M/S) \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{0.0.2}$$

becomes an isomorphism.

The idea behind the proof of this theorem is to combine the description of values of representable sheaves in the h-topologies, in terms of universal homeomorphisms, with an appropriate understanding of seminormality.

Structure of the thesis

Chapter 1: Preliminaries and generalities

We recall miscellaneous notions and results that will be needed at various points in the thesis.

Chapter 2: Relative cycles

In this second chapter we spend a lot of time recalling and familiarizing ourselves with the theory of relative cycles as developed in [SV00]. Our presentation of this material follows op.cit. rather closely, and sometimes adds some additional explanations (the table at the end of the chapter gives an overview of how our presentation compares to op.cit.). We also explain that Kollár's families of algebraic cycles satisfying the field of definition condition ([Kol96, Ch.I, Def. 4.7]) coincide with the relative proper cycles with universally integral coefficients, and we discuss the loci of points where relative cycles are effective and vanish (Section 2.7).

Chapter 3: h-topologies

In this chapter we recall Voevodsky's h-topology and some of its cousins. We discuss refinements of coverings in the h-topology by following and extending material found in [Voe96]. Furthermore we consider the presheaves of relative cycles in the context of the h-topologies essentially following [SV00]. We also briefly discuss some more general theory concerning sheaves in the qfh-topology. The final section of the chapter provides a table describing where we have extended results and/or expanded on proofs from the papers [Voe96], [SV96] and [SV00].

Chapter 4: Generalized seminormalization with applications

We introduce a construction whose purpose is to give a simple and uniform way to study properties of the twin theories of semi and weak normality. We state and prove several properties of our construction first for rings and then extend the construction to schemes. The final section of the chapter is an application to the study of representable sheaves in the h-topologies where we prove Theorem 4.3.9.

Much of this chapter was written in collaboration with Jarle Stavnes.

Chapter 5: Representable monoids in the h-topology

The sole purpose of this chapter is to prove Theorem 5.0.1 which is a key ingredient in the proofs of both our main theorems.

Chapter 6: Chow schemes and Chow monoids

The purpose of this chapter is to construct the Chow monoid of r dimensional cycles on a projective scheme and prove the first main theorem (Theorem 6.5.3).

The construction of the Chow monoid requires some intersection theory and the concept of relative effective Cartier divisors which we recapitulate. We also need to prove in a precise manner the fact that the presheaf of relative cycles of dimension r and degree d is representable in the h-topology as discussed in [SV00, Ch.4, Sec.4]. To achieve this latter task we explain and apply methodology from loc.cit. and [Kol96, Ch.I, Sec.3,4].

Chapter 7: Relative zero cycles via symmetric powers

In this final chapter we prove the second main theorem (Theorem 7.2.1) of the thesis.

A note to the reader

The author has written this thesis with the aim that it should be understandable to most algebraic geometers. For this reason chapters 1, 2, 6 and 7 all have (to varying extent) some overlap with [SV00]. Moreover chapters 3 and 7 will at times closely follow [Voe96]. Whenever a statement is similar to one found in the literature it will be emphasised, and in that case we do our best to explain how the proofs compare. In fact we have added a table at the end of Chapters 2, 3 and 6 that explains how our presentation of the material taken from the work of Suslin–Voevodsky (and other sources) compares to the original.

We have also included several appendices for the convenience of the reader not already familiar with one or more of these topics. The appendices differ from the first chapter in the sense that they will be referred to a lot less frequently and are therefore not as in depth.

By default we usually use the terminology and definitions of the stacks-project, and always try to let the reader know when this is not the case. Most of the time our schemes will be taken to be separated, but there are a few points where this is not necessary.

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[2]Paul Arne also deserves my apologies for not spending more time in the gym.

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[3]Award ref: 1556568

Chapter 1

Preliminaries and generalities

The purpose of this chapter is to collect general definitions and results, mostly from [GD67],[Bou64],[Bou03] and [Gro71], that are central to the theory of this thesis. We only provide proofs in the cases where we either did not find a reference in the literature, or when we think the reader not already familiar with the topic in question can pedagogically benefit from reading a proof.

On a first reading we recommend only reading Section 1.1, Section 1.2 and Section 1.7, the rest of the material in this chapter is not used as often and can be picked up when needed.

The statements of the chapter are either well known or easily deduced from such results.

1.1 Universally equidimensional morphisms

In this section we will assume all schemes to be separated to be on the safe side.

Dimension at a point

Definition 1.1.1 ([GD67, Ch.0, (14.1.2.)])**.** Let X be a topological space. For a given $x \in X$, we define the *dimension of* X *at* x denoted by $\dim_x(X)$ to be the number $\inf_U(\dim(U))$[1] where the infimum runs over the open neighborhoods of x.

Remark 1.1.2. Since the natural numbers are well ordered, we can find an open neighborhood U of x such that $\dim(U) = \dim_x(X)$.

We mention a few of the properties of the pointed dimension function $\dim_x$.

Proposition 1.1.3 ([GD67, (14.1.6)])**.** *For every topological space* X*, we have* $\dim(X) = \sup_{x \in X} \dim_x(X)$*.*

[1] In this thesis the number $\dim(U)$ denotes the combinatorial dimension of U

Proposition 1.1.4. *Suppose X is a Noetherian topological space satisfying the (T_0)-axiom and let F denote the set of closed points of X, then* $\dim(X) = \sup_{x \in F} \dim_x(X)$.

Proposition 1.1.5 ([GD67, (14.1.11)]). *The function $x \mapsto \dim_x(X)$ is upper semi-continuous.*

Lemma 1.1.6. *For an irreducible variety X we have that $\dim_x(X) = \dim(X)$ for every $x \in X$.*

Proof. This follows because if U is any nonempty open subset of X then $\dim(U) = \dim(X)$. □

Proposition 1.1.7. *Suppose that X is an algebraic scheme and U is any nonempty open subset of X. Let $X_1, \ldots, X_m$ denote the irreducible components of X meeting U. Then*

$$\dim U = \sup_{1 \leq i \leq m} \dim X_i.$$

Proof. The irreducible components of U are $X_1 \cap U, \ldots, X_m \cap U$ and by Lemma 1.1.6 we have that $\dim(X_i) = \dim U \cap X_i$ hence the result follows. □

Corollary 1.1.8. *Suppose that X is an algebraic scheme. Then for a point $x \in X$, let $X_1, \ldots, X_m$ denote the irreducible components of X containing the point x. Then we have*

$$\dim_x(X) = \sup_{1 \leq i \leq m} \dim X_i.$$

Corollary 1.1.9. *Suppose X is an equidimensional*[2] *algebraic scheme of dimension r, then for any open subscheme U we have $\dim(U) = \dim(X)$ and in particular we have $dim_x(X) = r$ for all $x \in X$.*

Proof. Follows immediately from Proposition 1.1.7. □

Definition 1.1.10. For a morphism $p : X \to S$ of schemes denote by $\dim(X/S)$ the function on the set of points of X of the form $\dim(X/S)(x) := \dim_x(p^{-1}(p(x)))$, which we shall call the *local fiber dimension function of X/S*.

Lemma 1.1.11. *Let $p : X \to S$ be a morphism of schemes. For any point $x \in X$ we have that $\dim_{X/S}(x)$ is at least the dimension of the closure of x in $p^{-1}(p(x))$.*

Proof. Follows immediately from Corollary 1.1.8. □

We recall the following version of Chevalley's theorem:

[2]As usual we say that a scheme or a topological space X is *equidimensional* or of *pure dimension* if for any pair of irreducible components X_1, X_2 of X we have $\dim(X_1) = \dim(X_2)$.

Theorem 1.1.12 ([GD67, Th. 13.1.3]). *(Chevalley) Let $p : X \to S$ be a morphism of finite type. Then for any $n \geq 0$ the subset $\{x \in X \mid \dim_{X/S}(x) \geq n\}$ is closed in X. In other words the function $\dim(X/S)$ is upper semi-continuous.*

Universally equidimensional morphisms

Definition 1.1.13 ([SV00, Def. 2.1.2]). A morphism of schemes $p : X \to S$ is called an *equidimensional* morphism of dimension r if the following conditions hold:

1. The morphism p is of finite type.
2. The function $\dim(X/S)$ is constant and equals r.
3. Any irreducible component of X dominates an irreducible component of S.

A morphism of schemes $p : X \to S$ is called *universally equidimensional* of dimension r if for any morphism $S' \to S$ the projection $X \times_S S' \to S'$ is equidimensional of dimension r.

Finally, we say that $p : X \to S$ is a *morphism of dimension* $\leq r$ if $\dim(X/S)(x) \leq r$ for all points x of X.

Proposition 1.1.14. *In the definition of an equidimensional morphism 1.1.13 condition 2 can be replaced with the condition that for any point x of X the scheme $p^{-1}(p(x))$ is equidimensional of dimension r.*

Proof. If the function $\dim(X/S)$ is constant and equal to r, then by definition we have

$$\dim p^{-1}(p(x)) \geq \dim(X/S)(x) = r.$$

If η is a generic point of $Y = p^{-1}(p(x))$ then $\dim(X/S)(\eta) = r$ and we can find an open subset U of η not containing the other generic points of Y such that $\dim(U) = r$ hence we have that any irreducible component of Y has dimension at least r. On the other hand if Y_i is an irreducible component of Y and $C_0 \subsetneq \ldots \subsetneq C_m = Y_i$ is a maximal chain of closed irreducible subsets of Y_i, then letting z be the generic point of C_0 we can find some open neighborhood U of the point z of dimension r, but then we must have $m \leq r$ hence every irreducible component of Y has dimension at most r as well.

The converse statement follows immediately from Corollary 1.1.9. □

Lemma 1.1.15. *Conditions 1 and 2 in Definition 1.1.13 are both stable under base change.*

Proof. It is well known that finite type morphisms are stable under basechange. For Item 2 see [Stacks, Tag 02FY]. □

Lemma 1.1.15 does not imply that every equidimensional morphism is universally equidimensional. Indeed Item 3 of Definition 1.1.13 is not necessarily stable under base change:

Example 1.1.16. Let L_1, L_2 be two lines in the affine plane intersecting in a point. Let $S = L_1 \cup L_2$ and note that the inclusions of L_1, L_2 in S induce a morphism $X = L_1 \coprod L_2 \to S$. This morphism is obviously equidimensional, but the base change $L_1 \times_S X = (L_1 \cap L_1) \coprod (L_1 \cap L_2) \to L_1$ is clearly not equidimensional.

We give some examples and counter examples to morphisms which are equidimensional.

Example 1.1.17.

1. Those familiar with Pseudo-Galois coverings (Definition 2.5.11) will see from [SV96, Corollary 5.10] that such morphisms are universal equidimensional morphisms of dimension 0 when the base scheme is normal.

2. The vector bundle $p : B \to X$ induced by a finite type quasi-coherent sheaf $\mathcal{F}$ of (constant) rank n on a scheme X is a universal equidimensional morphism of dimension n. Indeed this follows easily from the fact that the pullback of $\mathcal{F}$ to X_{red} must necessarily be locally free of rank n.

3. A simple example of a morphism whose fibers are not equidimensional is to consider two lines L_1, L_2 intersecting at a point and then mapping this scheme to L_1 and mapping the entire L_2 to the point of intersection. Consider for instance $V(y) \cup V(x) = \mathrm{Spec}(k[x, y]/(xy)) \to \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ given by $t \mapsto x$.

4. Another nice example of a morphism which fails to be equidimensional would be to consider the map $p : \mathbb{A}^2_k \to \mathbb{A}^2_k$ induced by $x \mapsto x^2,\ y \mapsto (1 - x)y$. The fiber
$$p^{-1}(p(1, 0)) = p^{-1}((1, 0))$$
has two irreducible components one being a line and the other a point, hence p is not an equidimensional morphism [3].

The next proposition shows that Item 2 of Example 1.1.17 is not so far from the only equidimensional morphisms.

Proposition 1.1.18 ([SV00, Prop.2.1.3])**.** *Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. Then p is equidimensional of dimension r if*

[3] This last example was pointed out to the author by Remy van Dobben de Bruyn.

and only if for any point x of X there is an open neighborhood U in X and a factorization of the morphism $p_U : U \to S$ of the form

$$U \overset{p_0}{\to} \mathbb{A}^r_S \to S$$

such that p_0 is a quasi-finite morphism and any irreducible component of U dominates an irreducible component of $\mathbb{A}^r_S$.

Proof. See [GD67, 13.3.1(b)] $\square$

Corollary 1.1.19. *Let $p : X \to S$ be an equidimensional morphism of dimension r. If X' is any irreducible component of X then the composition $X' \to X \overset{p}{\to} S$ is also an equidimensional morphism of dimension r.*

Proof. Let x be a point of X', and let U be an open neighborhood of x in X such that we have a factorization of the morphism $p_U : U \to S$ of the form $U \overset{p_0}{\to} \mathbb{A}^r_S \to S$ as in Proposition 1.1.18. Then $U \times_X X' \overset{p_0}{\to} \mathbb{A}^r_S \to S$ is a factorization of $U \times_X X' \to X' \to X \overset{p}{\to} S$ and it is clear that $U \times_X X'$ has only got one irreducible component corresponding to the generic point of X', thus it must dominate an irreducible component of $\mathbb{A}^r_S$. $\square$

In this thesis we will occasionally work with so called (geometrically) unibranched schemes. The reader not familiar with this notion is encouraged to consult Chapter C or simply replace it with the special case of normal schemes throughout.

Proposition 1.1.20 ([SV00, Prop.2.1.7])**.** *Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. The following implications hold:*

1. *If p is a universally equidimensional morphism, then p is universally open.*

2. *If $\dim(X/S) = r$ and p is open (resp. universally open) then p is equidimensional (resp. universally equidimensional) of dimension r.*

3. *If S is geometrically unibranch and p is equidimensional then p is universally equidimensional (and hence universally open).*

Proof. A clear proof can be found in [SV00] $\square$

Corollary 1.1.21 ([SV00, p.8, Rmk 3])**.** *Let $p : X \to \mathrm{Spec}(k)$ be a morphism of finite type. Then p is universally equidimensional of dimension r if and only if all irreducible components of X have dimension r.*

Proof. Follows immediately from Proposition 1.1.20 Item 3. $\square$

The following results tell us how to check that if a given flat morphism is of relative dimension r.

Lemma 1.1.22 ([Stacks, Tag 0D4H]). *Let $f : X \to Y$ be a flat morphism of schemes of finite presentation. Let $n_{X/Y} : Y \to \mathbb{N}$ be the function on Y giving the dimension of fibres of f given by $y \mapsto \dim X_y$. Then $n_{X/Y}$ is lower semi-continuous.*

Proposition 1.1.23 ([SV00, Prop. 2.1.8]). *Let $p : X \to S$ be a flat morphism of finite type with S a Noetherian scheme. Then p is universally equidimensional of dimension r if and only if for any generic point $y : \mathrm{Spec}(K) \to S$ of S the projection $X \times_S \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ is equidimensional of dimension r (or the fiber is empty).*

Proof. The proof that we shall give here was pointed out to the author by Remy van Dobben de Bruyn.

Note that one direction is trivial. For the other note that by universal openness of flat morphisms of finite presentation ([Stacks, Tag 01UA]) and our assumptions we have that the scheme $X_{p(g)}$ is equidimensional of dimension r for every generic point g of X. Thus by Corollary 1.1.9 it follows that $\dim(X/S)(g) = r$ for every generic point g of X hence by Chevalley's theorem 1.1.12 we conclude the equality of sets

$$X = \{x \in X \mid \dim(X/S)(x) \geq r\}$$

Suppose now for the sake of contradiction that there is some $x \in X$ such that $\dim(X/S)(x) > r$. Then we must have

$$\dim X_{p(x)} > r$$

and $p(x)$ is contained in the set $U = \{y \in Y \ : \ \dim X_y > r\}$ which by Lemma 1.1.22 is open. But then $p^{-1}(U)$ is a non-empty open subset of X thus contains at least one generic point of X giving a contradiction. □

Proposition 1.1.24 ([SV00, Proposition 2.1.9]). *Suppose $p : X \to S$ is an equidimensional morphism of relative dimension r such that X is irreducible. Suppose that p admits a decomposition of the form*

$$X \xrightarrow{p_0} W \xrightarrow{p_1} S$$

with p_0 surjective and proper and p_1 has at least one fiber of dimension r. Then p_1 is equidimensional of dimension r and p_0 is finite in the generic point of W.

Lemma 1.1.25 ([SV00, Lemma 2.1.10]). *Let $p : X \to S$ be a morphism such that any irreducible component of X dominates an irreducible component of S and $i : X_0 \to X$ be a closed embedding which is an isomorphism over the generic points of S. Then i is defined by a nilpotent sheaf of ideals. In particular, p is a universally equidimensional morphism of dimension r if and only if $p_0 : X_0 \to S$ is a universally equidimensional morphism of dimension r.*

Corollary 1.1.26 ([SV00, Lemma 2.1.11]). *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $Z \subset X$ be a closed subscheme universally equidimensional of relative dimension r over S and $S' \to S$ be a blow-up of S. Let $\tilde{Z}$ be the proper transform of Z in $X \times_S S'$. Then $\tilde{Z}$ is a closed subscheme of $Z \times_S S'$ defined by a nilpotent sheaf of ideals.*

Proof. Since $Z \times_S S'$ is equidimensional over S' and hence its generic points lie over generic points of S' our statement follows from Lemma 1.1.25. $\square$

1.2 Flatification by blowing-up

The flatification theorem from [RG71] plays a crucial role in the theory of relative cycles. Before stating the theorem we briefly recall the proper transform. We keep the assumption that our schemes are separated throughout this section.

Definition 1.2.1. Let Z be a closed subscheme of a scheme S. Let $p : S' \to S$ be the blowup of S with center in Z. For a scheme $Y \to S$ over S, denote by $\tilde{Y}$ the scheme theoretic closure in $Y \times_S S'$ of the open subscheme $Y \times_S S' \setminus pr_2^{-1}(p^{-1}(Z))$. The scheme $\tilde{Y}$ is called the *strict transform* or *proper transform* of Y with respect to p.

Lemma 1.2.2. *Suppose that S is a scheme and $f : Y \to S$ be any scheme over S. For any closed subscheme Z of S let $p : S' \to S$ be the blow-up of S with center Z. Then the proper transform*

$$\tilde{Y} \hookrightarrow Y \times_S S'$$

is an isomorphism over the open subscheme $S' \setminus p^{-1}(Z)$. Hence an isomorphism over all the generic points of S'.

Proof. Let $i : Y \times_S S' \setminus pr_2^{-1}(p^{-1}(Z)) \to Y \times_S S'$ denote the open embedding. We have that the proper transform $\tilde{Y}$ is cut out by a subsheaf of

$$\operatorname{Ker}\left(i^{\#} : \mathcal{O}_{Y \times_S S'} \to i_* \mathcal{O}_{Y \times_S S' \setminus pr_2^{-1}(p^{-1}(Z))}\right)$$

which obviously vanishes over $S' \setminus p^{-1}(Z)$. The last statement because $p^{-1}(Z)$ is an effective Cartier divisor and thus contains no generic points of S'. $\square$

Theorem 1.2.3 (Flatification by blowup). *Let $f : X \to S$ be a morphism of finite type of Noetherian schemes, which is flat over an open subset $U \subset S$. Then there exists a closed subscheme $Z \subset S$ disjoint with U such that if S' denotes the blow up of S in Z then the proper transform $\tilde{X}$ is flat over S'.*

Proof. This is the statement of [Voe96, Theorem 3.1.8]. For a proof see [RG71, Sec. 5.2] where the result originates from, or one can see [Stacks, Tag 081R]. $\square$

1.3 Valuation rings and valuative criteria

Valuation rings

Let us briefly recall the basic theory of valuation rings.

Definition 1.3.1. Valuation rings.

1. Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.

2. We say that a local ring R is a *valuation ring* if R is an integral domain and maximal with respect to the order relation of domination among local rings contained in the fraction field of R.

3. Let R be a valuation ring with field of fractions K. If $A \subset K$ is a subring of K, then we say that R is *centered* on A if $A \subset R$.

Lemma 1.3.2. *Let K be a field. Let $A \subset K$ be a local subring. Then there exists a valuation ring with fraction field K dominating A.*

Proof. See [Stacks, Tag 00IA]. □

A proof of the following Lemma is given in [Stacks, Tag 00IB].

Lemma 1.3.3. *Let R be an integral domain with field of fractions K. Then R is a valuation ring if and only if for any $x \in K$ we either have $x \in R$ or $x^{-1} \in R$ (or both).*

Lemma 1.3.4. *Let A be a ring. Then every specialization $p_1 \subseteq p_2$ is covered by some homomorphism $A \to R$, where R is a valuation ring. Moreover, the valuation ring R can be chosen such that its field of fractions is $k(p_1)$.*

Proof. Consider the homomorphism $g : A \to (A/p_1)_{p_2}$, and let (R, m_R) be a valuation ring dominating $(A/p_1)_{p_2}$ in its field of fractions (Lemma 1.3.2). Consider the inclusion $f : (A/p_1)_{p_2} \to R$. Then $f^{-1}(m_R) = p_2(A/p_1)_{p_2}$ and $f^{-1}(0) = (0)$. Consider the homomorphism $h = f \circ g : A \to R$. Then $h^{-1}(m_R) = p_2$ and $h^{-1}(0) = p_1$. In other words h covers the specialization $p_1 \subseteq p_2$. □

Lemma 1.3.5. *Let R be a valuation ring. For any prime ideal $\mathfrak{p} \subset R$, the quotient $R/\mathfrak{p}$ is a valuation ring. Furthermore, any localization of R is again a valuation ring.*

Proof. See [Stacks, Tag 088Y]. □

Lemma 1.3.6. *Any valuation ring R is normal.*

Proof. A proof is given in [Stacks, Tag 00IC]. □

Lemma 1.3.7. *Let $R \to R'$ be an inclusion of valuation rings, and assume that the induced morphism* $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ *is surjective. Then*

$$R' \cap R_{(0)} = R \subset R'_{(0)}. \tag{1.3.1}$$

Proof. Suppose that $x \in R' \cap R_{(0)}$. If x is not contained in R, then $x^{-1} \in R$, which means that x is a unit in R'. Since the inverse image of the maximal ideal of R' must necessarily be the maximal ideal of R it follows then that $x^{-1} \in R$ cannot be contained in the maximal ideal of R. Thus x^{-1} is a unit in R hence we have $x \in R$ giving a contradiction. □

Lemma 1.3.8. *Let R be a valuation ring then every R-module without torsion is flat over R. In particular if $R \to A$ is an injective ring homomorphism to an integral domain then A is flat as an R-module.*

Proof. See [Bou64, Ch. VI, Sec. 3.6, Lemma 1, Page 106]. □

Recall that a valuation ring which is not a field is said to be *discrete* if it is Noetherian which is equivalent to being a local Noetherian normal domain of dimension 1 and also equivalent to being a local Noetherian domain whose maximal ideal is generated by a single nonzero element.

Proposition 1.3.9. *Let R be a valuation ring with field of fractions K and let L/K be any field extension. Then there exists an injection $R \to R'$ where R' is a valuation ring with field of fractions L such that the induced map* $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ *is faithfully flat. If furthermore R is a discrete valuation ring and L/K is a finite extension then the valuation ring R' can be taken to be discrete.*

Proof. By Lemma 1.3.2 we can find a valuation ring R' of L dominating R and by Lemma 1.3.8 the map $R \to R'$ is a flat local homomorphism hence it is faithfully flat. The final statement follows now directly from Krull-Akizuki [Stacks, Tag 00PG]. □

Remark 1.3.10. It is possible to prove Proposition 1.3.9 in a (potentially) more constructive way. Indeed if we first consider the case where the extension L/K is algebraic then letting S denote the integral closure of R in L and $\mathfrak{m}$ any maximal ideal of S. Set $R' = S_{\mathfrak{m}}$. By[Bou64, Ch.6, Sec. 6, Prop. 6, Page 147] R' is a valuation ring and it follows easily from the going up and down theorems that the induced map $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ is surjective.

In the case when L/K is a purely transcendental extension say $L = K(\{t_i\}_{i \in I})$ where the t_i are independent variables. Let $v : K^\times \to \Gamma$ be the valuation corresponding to R. For a polynomial $P \in K[\{t_i\}_{i \in I}]$ set

$$w(P) := \min\{v(c) \mid c \text{ is a coefficient of } P.\} \tag{1.3.2}$$

Then one can show that this induces a well defined valuation w of L with value group Γ such that w extends v. Thus letting R' denote the valuation ring corresponding to w we get an inclusion $R \subset R'$ with the induced map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ being surjective.

Now for the general case recall that any field extension L/K has some transcendental basis hence we get a tower of field extensions $K \subset L_0 \subset L$ where L_0/K is purely transcendental and L/L_0 is algebraic. By combining the two steps described above we obtain the desired valuation ring R'.

Valuative criteria

Recall the valuative criterion of universal closedness which we will apply on several occasions.

Proposition 1.3.11. *(Valuative criterion of universal closedness) Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion*[4].

Proof. See [Stacks, Tag 01KF]. □

Lemma 1.3.12. *Let $p : S' \to S$ be a quasi-compact, surjective and universally closed morphism of schemes. Let R be a valuation ring, and let $g : \mathrm{Spec}(R) \to S$ be some morphism, where R is a valuation ring. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(R') & \xrightarrow{g'} & S' \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle p} \\ \mathrm{Spec}(R) & \xrightarrow{g} & S \end{array} \tag{1.3.3}$$

such that R' is a valuation ring and h is faithfully flat. Moreover, R' can be chosen such that $R' \cap R_{(0)} = R$, where the intersection is taken inside the field of fractions of R'. Further still if R is a discrete valuation ring and p is also assumed to be of finite type then R' may be chosen such that it is a discrete valuation ring.

Proof. Let K denote the field of fractions of R. Since p is surjective we can find some field extension L/K and a map $j : \mathrm{Spec}(L) \to S'$ such that the compositions of morphisms

$$\mathrm{Spec}(L) \to \mathrm{Spec}(K) \to \mathrm{Spec}(R) \to S$$

coincides with the composition $p \circ j$. By Proposition 1.3.9 we get an inclusion of valuation rings $R \to R'$ where L is the field of fractions of R' and the induced map $h : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective. The existence of the desired map g' follows now from the valuative criterion of universal closedness (1.3.11).

It follows from Lemma 1.3.7 that R' can be chosen such that $R' \cap R_{(0)} = R$.

For the final statement note that if p is of finite type then the field extension L/K can be taken to be finite. □

[4]See [Stacks, Tag 01KD].

Remark 1.3.13. The final statement of this lemma is [SV00, Lemma 3.3.4]. More generally [Ryd10, Corollary 2.9] states that if p is a quasi-compact morphism then there exists a diagram of the form (1.3.3) if and only if p is *universally subtrusive*. We will not define this notion here, but we let the reader know there are many examples of subtrusive morphisms such as for instance any surjective universally open morphism; See [Ryd10, Remark 2.5] for more examples.

There is also a valuative criterion for flatness, which will only be used at one point in this thesis.

Theorem 1.3.14 (Valuative criterion for flatness)**.** *Let $f : X \to Y$ be a morphism locally of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation, x a point of X and $y = f(x) \in Y$. Suppose that the local ring $\mathcal{O}_{Y,y}$ is integral (resp. reduced and Noetherian). For $\mathcal{F}$ to be f-flat at x it is sufficient and neccesary that for any valuation ring (resp. discrete valuation ring) A' and local ring homomorphism $\mathcal{O}_{Y,y} \to A'$ the following condition is satisfied: Setting $Y' = \mathrm{Spec}(A')$, $X' = X \times_Y \mathrm{Spec}(A')$ and $f' = X' \to Y'$ the projection, the $O_{X'}$ module $\mathcal{F}' = \mathcal{F} \otimes_Y Y'$ is f'-flat at every point x' of X' mapping to $x \in X$ and the closed point $y' \in Y'$ under the two projections $X' \to X$ and $X' \to Y'$ respectively.*

Proof. This is [GD67, (11.8.1)]. □

1.4 Fields and related notions

Integral closure in extensions of total rings of fractions

We recollect a bunch of results concerning integral closure in extensions of fields of fractions that will implicitly be used on several occasions.

Definition 1.4.1. For a ring A let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called the *total ring of fractions* of A.

Lemma 1.4.2. *Let R be a reduced ring. Then*

1. *R is a subring of a product of fields,*
2. *$R \to \prod_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$ is an embedding into a product of fields,*
3. *$\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$ is the set of zerodivisors of R.*

Proof. See [Stacks, Tag 00EW] □

The following Lemma is proved in [Stacks, Tag 02LX].

Lemma 1.4.3. *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Lemma 1.4.4. *Let X be a reduced scheme such that every quasi-compact open subset has finitely many irreducible components. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then*

1. *A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,*
2. *the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,*
3. *the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and*
4. *$\nu^{-1}(U)$ is identified with the spectrum of A' where $\nu : X^\nu \to X$ is the normalization morphism.*

Proof. See [Stacks, Tag 035P]. □

Lemma 1.4.5. *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S_i'. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S_i'. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.*

Proof. A proof can be found in [Stacks, Tag 0CY9]. □

Lemma 1.4.6. *Let A be a reduced ring with finitely many minimal prime ideals say $p_1, \ldots, p_t$. For each such minimal prime ideal p_i suppose that L_i is a field extension of the field $k(p_i) = A_{p_i}$. For each i let $\overline{A_i}$ (resp. $\tilde{A}_i$) denote the integral closure of $A \to k(p_i)$ (resp. of $A \to k(p_i) \to L_i$).*

The integral closure $\tilde{A}$ of A in $A \to \prod_{i=1}^t k(p_i) \to \prod_{i=1}^t L_i$ is the product $\prod_i \tilde{A}_i$. Hence there is a one to one correspondence between minimal primes of $\tilde{A}$ and those of A. Moreover if all the extensions $L_i/k(p_i)$ are algebraic then the field of fractions of $\tilde{A}_i$ coincides with L_i and the total ring of fractions $Q(\tilde{A})$ of $\tilde{A}$ is the product $\prod_i L_i$.

Proof. Letting $\overline{A} = \prod_{i=1}^t \overline{A_i}$ denote the integral closure of A in $Q(A)$ we have by Lemma 1.4.5 that the integral closure of $\overline{A}$ (and hence of A) in $\prod_{i=1}^t L_i$ is the product $\prod_{i=1}^t \tilde{A}_i$. □

Lemma 1.4.7. *Let A, A' be reduced rings and $A \to A'$ a faithfully flat ring homomorphism. Let $\overline{A}$ denote the integral closure of A in its total ring of fractions $Q(A)$. Then A' and $A' \otimes_A \overline{A}$ have the same total ring of fractions.*

Proof. Let $S = A \setminus \cup_{p \text{ minimal}} p$ denote the set of non-zero divisors of A. Since flat ring homomorphisms satisfy the goingdown property we easily see that if $s \in S$ then the image of s in A' cannot be a zero divisor of A', hence $Q(A') = Q(S^{-1}A')$. Now consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & \overline{A} \arrow[r, hook] \arrow[d] & S^{-1}A = Q(A) \arrow[d] \\ A' \arrow[r] & (A' \otimes_A \overline{A}) \arrow[r] & S^{-1}A'. \end{tikzcd} \tag{1.4.1}$$

By faithful flatness it follows that all horizontal arrows are injections ([Stacks, Tag 00HO]) and one easily checks that each of these horizontal arrows has the property that non-zero divisors are mapped to non-zero divisors. Thus we get induced injections

$$Q(A') \hookrightarrow Q(A' \otimes_A \overline{A}) \hookrightarrow Q(S^{-1}A') = Q(A') \tag{1.4.2}$$

and since the composition of these maps is an isomorphism we conclude that the map $Q(A') \to Q(A' \otimes_A \overline{A})$ is also an isomorphism. $\square$

Separable extensions

We recall the basic theory of separable field extensions.

Recall that an algebraic field extension L/K is *separable* if for any element $x \in L$ the corresponding minimal polynomial is relatively prime to its derivative.

The following definition found in (for instance) [Stacks, Tag 030O] defines separability for arbitrary field extensions.

Definition 1.4.8. Let $k \subset K$ be a field extension.

1. We say K is *separably generated* over k if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the algebraic extension $k(x_i; i \in I) \subset K$ is separable.

2. We say K is *separable* over k if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension $k \subset K'$ is separably generated.

Note that if k is a field of characteristic zero then any field extension of k is necessarily separable.

Lemma 1.4.9. *Let k be a field. Let S be a reduced k-algebra. Let $k \subset K$ be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.*

Proof. See [Stacks, Tag 030U]. $\square$

As suggested by Lemma 1.4.9 separable field extensions are closely related to the notion of geometrically reduced algebras which we now recall in the following definition:

Definition 1.4.10. Let k be a field. Let S be a k-algebra. We say S is *geometrically reduced* over k if for every field extension $k \subset K$ the K-algebra $K \otimes_k S$ is reduced.

Lemma 1.4.11. *Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k.*

Proof. See [Stacks, Tag 04KN]. □

Lemma 1.4.12. *Let $k \subset K$ be a field extension. Then K is separable over k if and only if K is geometrically reduced over k.*

Proof. One implication follows directly from Lemma 1.4.9. For the other see [Stacks, Tag 030W]. □

Lemma 1.4.13. *Let $k \subset K' \subset K$ be a tower of field extensions and suppose that K'/k and K/K' are separable. Then K/k is separable.*

Proof. Follows easily from Lemma 1.4.12 and Lemma 1.4.9. □

Lemma 1.4.14. *Let $k \subset K$ be a separable extension and let $k \subset L$ be any field extension. Then if $\mathfrak{p}$ is any minimal prime ideal of $S = L \otimes_k K$ then $E = S_{\mathfrak{p}}$ is a field and the canonical map $L \to E$ is a separable field extension.*

Proof. It follows from Lemma 1.4.12 that S is a geometrically reduced L-algebra. By Lemma 1.4.11 it follows that E is a geometrically reduced L-algebra and since E is reduced with exactly one prime ideal it is necessarily a field. By Lemma 1.4.12 again we conclude. □

In order to be self contained we also recall the definition of a perfect field.

Definition 1.4.15. Let k be a field. We say k is *perfect* if every field extension of k is separable over k.

Lemma 1.4.16. *Let k be a field.*

1. *If k is of characteristic zero then k is perfect.*
2. *if k is of characteristic $p > 0$ then k is perfect if and only if every element of k has a p'th root in k.*

Proof. See [Stacks, Tag 030Z]. □

Purely inseparable extensions

Basics

Recall that the *exponential characteristic* of a field is defined to be 1 if it is of characteristic zero, otherwise it is the same number as the characteristic of the field. For a field k of exponenital characteristic p and a field extension K/k we recall that an element $x \in K$ is *purely inseparable* over k if there exists a power q of p such that $x^q \in k$. The field extension K/k is said to be *purely inseparable*[5] if every element $x \in K$ is purely inseparable over k. We summarise the basic properties of such extensions here:

Proposition 1.4.17 ([Bou03, Ch. V., Sec. 5.1])**.** *Let k be a field of exponential characteristic p and K/k a field extension. The following statements hold true:*

1. *If $x \in K$ is purely inseparable over k and e is the smallest positive integer such that $x^{p^e} \in k$ then the minimal polynomial of x over k is $P(X) = X^{p^e} - x^{p^e}$ and we have $[k(x) : k] = p^e$.*

2. *Let $(K)_{pi}$ denote the subset of K consisting of those elements which are purely inseparable over k. Then $(K)_{pi}$ is a subextension of K/k and it is the largest purely inseparable extension of k contained in K.*

3. *Suppose that the extension K/k is purely inseparable. Then given a morphism $u : k \to E$ where E is a perfect field, there is a unique map $v : K \to E$ extending u.*

4. *Suppose the field extension K/k is finite and purely inseparable. Then the degree $[K : k]$ is a power of p.*

Definition 1.4.18. For an field extension K/k we define the *purely inseparable closure* of k in K, denoted $(K)_{pi}$, to be the field extension found in Item 2 of Proposition 1.4.17.

Lemma 1.4.19 ([Stacks, Tag 030K])**.** *Let K/k be an algebraic field extension. There exists a unique sub extension $k \subset K_{\text{sep}} \subset K$ such that K_{sep}/k is separable and K/K_{sep} is purely inseparable.*

Definition 1.4.20. For a field extension K/k we will call the extension K_{sep} found in Lemma 1.4.19 the *separable closure of* k in K. Furthermore if K/k is a finite extension we also introduce the following numbers:

1. The integer $[K_{\text{sep}} : k]$ is called the *separable degree* of the extension K/k we will denote this integer by $[K : k]_{\text{sep}}$.

2. The integer $[K : K_{\text{sep}}]$ is called the *inseparable degree* and we denote it by $[K : k]_{\text{insep}}$.

[5] This also goes by the name *p-radical* in the literature.

Lemma 1.4.19 tells us that any algebraic extension can be decomposed into a separable extension followed by a purely inseparable one. The opposite is possible in the case of a normal extension:

Lemma 1.4.21 ([Rom06, Thm.3.6.4]). *Let K/k be a normal extension. Then the subfield $(K)_{pi}$ coincides with the subfield K^G of elements invariant under the action of $G = \mathrm{Gal}(K/k)$. Furthermore in the tower*

$$k \subset (K)_{pi} \subset K \tag{1.4.3}$$

the first step is purely inseparable and the second step is separable.

Other characterizations

In Chapter 4 we will apply a few other equivalent characterizations of purely inseparable extensions which we now will start to recall. To this extent we will first need a standard result in descent theory.

Lemma 1.4.22. *Let $f : A \to A', g : A \to B$ be ring homomorphisms with f faithfully flat. Then the following statements hold true:*

1. *The induced map $A' \to A' \otimes_A B$ is injective if and only if $g : A \to B$ is.*
2. *If the equivalent conditions of 1 are satisfied then the following pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A' \otimes_A B \end{tikzcd} \tag{1.4.4}$$

is also a pullback diagram.

Proof. The first statement follows easily from [Stacks, Tag 00HO]. For the second it is enough to show that if $a' \in A'$ and $b \in B$ satisfy the equality

$$a' \otimes_A 1 = 1 \otimes_A b \tag{1.4.5}$$

then there exists (necessarily unique) $a \in A$ such that $a' = f(a), b = g(a)$. By flatness of f we get a canonical isomorphism of A-modules

$$A' \otimes_A (B/A) \cong A' \otimes_A B/(A' \otimes_A A) \tag{1.4.6}$$

and by applying [Stacks, Tag 00HO] we get that the map

$$(B/A) \to (A' \otimes_A (B/A)) \cong A' \otimes_A B/(A' \otimes_A A) \tag{1.4.7}$$

given by $b \mapsto b \otimes 1$ is injective. Thus if $a' \otimes_A 1 = 1 \otimes_A b$ is satisfied then the image of b in B/A is zero hence there is some $a \in A$ with $b = g(a)$ and we must necessarily have $a' = f(a)$. $\square$

The next lemma may be considered a baby case of Manaresi's characterisation of weak normalization:

Lemma 1.4.23. *Let K/k be a field extension. The following diagram is an equalizer*

$$(K)_{pi} \hookrightarrow K \underset{(1\otimes)_{red}}{\overset{(\otimes 1)_{red}}{\rightrightarrows}} (K \otimes_k K)_{red} \tag{1.4.8}$$

Proof. Suppose that $x \in K$ is purely inseparable over k. Then if p denotes the exponential characteristic of K we have some $q = p^n$ for $n \geq 1$ such that $(x \otimes 1 - 1 \otimes x)^q = 0 \in K \otimes_k K$ hence

$$(x \otimes 1)_{red} = (1 \otimes x)_{red}. \tag{1.4.9}$$

To finish the proof it is enough to show that if $x \in K$ satisfies $(x \otimes 1)_{red} = (1 \otimes x)_{red}$ then $x \in (K)_{pi}$. Indeed suppose the last equality holds then we can find some N such that $(x \otimes 1 - 1 \otimes x)^N = 0$ and we may assume that $N = p^n$ for some $n \geq 1$. Thus $x^N \otimes 1 = 1 \otimes x^N \in K \otimes_k K$. By Lemma 1.4.22 we conclude that $x^N \in k$. □

We will also need the following characterisation of purely inseparable extensions which is a rather special case of [GD71, Proposition (3.7.1)] (or Lemma 1.4.28 which we will recall in the next subsection).

Lemma 1.4.24. *Let K/k be a field extension. The following are equivalent:*

1. *The extension K/k is purely inseparable.*

2. *Given any field extension E/k there is at most one k-algebra extension of K to E.*

Proof. Clearly (1) implies (2). Conversely if (2) holds then for any $x \in K$ we must necessarily have that $x \otimes 1, 1 \otimes x \in K \otimes_k K$ have the same image in every residue field of $K \otimes_k K$ hence by Lemma 1.4.23 we conclude that K/k is a purely inseparable extension. □

The above considerations lets us prove the following lemma which will play a starring role in Chapter 4 (and not needed before that).

Lemma 1.4.25. *Let F be an endofunctor on the category of fields and $\eta : Id_{fields} \to F$ be a natural transformation. The following are equivalent:*

1. *The field extension $\eta(k) : k \to F(k)$ is purely inseparable for all fields k.*

2. *The morphisms $F(\eta(k)) : F(k) \to F(F(k))$ and $\eta(F(k)) : F(k) \to F(F(k))$ coincide for all fields k.*

Proof. We always have

$$F(\eta(k)) \circ \eta(k) = \eta(F(k)) \circ \eta(k) \tag{1.4.10}$$

Thus if (1) holds then clearly (2) holds as well. Conversely if (2) holds then if $t_1, t_2 : F(k) \to K$ are two k-algebra homomorphisms to a field K such that $t_1 \circ \eta(k) = t_2 \circ \eta(k)$ it follows that

$$\eta(K) \circ t_i = F(t_i) \circ \eta(F(k)) = F(t_i \circ \eta(k)) \tag{1.4.11}$$

for $i = 1, 2$. Hence $\eta(K) \circ t_1 = \eta(K) \circ t_2$ thus $t_1 = t_2$. □

Example 1.4.26 (Perfect closure)**.** Let k be a field. There exists a perfect field k^{Perf} called the *perfect closure* of k and a field extension $u_k : k \to k^{Perf}$ such that if $v : k \to K$ is any other field extension to a perfect field then there exists a unique morphism $h : k^{Perf} \to K$ such that $v = h \circ u_k$

We sketch the construction of k^{Perf} here: If k is a field of characteristic 0 then set $k^{Perf} = k$ and let u_k be the identity map. Otherwise if k is of characteristic $p > 0$ we proceed as follows: For each $n \in \mathbb{N}$ set $k_n := k$ and for $m \geq n$ let $i_{n,m} : k_n \to k_m$ be the map given by $x \mapsto x^{p^{m-n}}$ and let k^{Perf} be the (directed) colimit in the category of rings of this diagram. Let u_k be the canonical map from $k = k_0$ to the colimit k^{Perf}. To see that the ring k^{Perf} is in fact a field and satisfies the relevant universal property see [Bou03, Ch. V, Sec.1.4, Thm 3] and [Bou03, Ch. V, Sec.1.4, Prop. 3].

Note that the assignment $k \mapsto k^{Perf}$ is functorial and the morphism $u_k : k \to k^{Perf}$ is natural in k. We will denote this natural transformation by $(-)^{Perf}$ throughout.

Radicial morphisms

Recall the following definitions.

Definition 1.4.27. Let $f : X \to S$ be a morphism of schemes.

1. We say f is *universally injective* if for any morphism of schemes $S' \to S$ the base change $f' : S' \times_S X \to S'$ is injective (on the underlying topological spaces).

2. We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the induced map of residue fields $\overline{f}_{x/f(x)} : k(f(x)) \to k(x)$ is purely inseparable.

3. We say f is a *universal homeomorphism* if for any morphism of schemes $S' \to S$ the base change $S' \times_S X \to S'$ is a homeomorphism of underlying topological spaces.

As mentioned earlier Lemma 1.4.24 is a special case of the following:

Lemma 1.4.28. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *For every field K the induced map* $\mathrm{Hom}(\mathrm{Spec}(K), X) \to \mathrm{Hom}(\mathrm{Spec}(K), S)$ *is injective.*

2. *The morphism f is universally injective.*

3. *The morphism f is radicial.*

4. *The diagonal morphism $\Delta_{X/S} : X \to X \times_S X$[6] is surjective.*

Proof. See [Stacks, Tag 01S4]. □

Corollary 1.4.29. *Let $f : X \to S$ be a radicial morphism of schemes and $g_1, g_2 : Y \to X$ be any two morphisms of schemes such that $f \circ g_1 = f \circ g_2$. If Y is reduced then $g_1 = g_2$.*

Proof. From Lemma 1.4.28 Part 1 it follows that for any field K and morphism $\tau : \mathrm{Spec}(K) \to Y$ we have $g_1 \circ \tau = g_2 \circ \tau$. From the fact that Y is reduced we can now conclude that $g_1 = g_2$ ([Stacks, Tag 01KM]). □

It is also useful to recall the following characterisation of universal homeomorphisms.

Lemma 1.4.30. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent.*

1. *f is a universal homeomorphism.*

2. *f is integral, universally injective and surjective.*

Proof. See [Stacks, Tag 04DF]. □

1.5 Group actions and quotients

Group actions on categories

We know that giving a left group action on a set X is the same as giving a group homomorphism

$$\rho : G \to \mathrm{Aut}(X)$$

and a right action is equivalent to group homomorphism from G^{op} . Since we can take the automorphism group of any object in a category we can define a group action of an object in a category completely analogous to the case of sets:

[6] As in the literature the diagonal morphism $\Delta_{X/S}$ (which is also sometimes denoted by $\delta_{X/S}$) is defined as the unique morphism satisfying $pr_1 \circ \Delta_{X/S} = pr_2 \circ \Delta_{X/S} = id_X$.

Definition 1.5.1. Let X be an object in a category $\mathscr{C}$. A *(left) group action on* X is a group homomorphism [7]

$$\rho : G \to \mathrm{Aut}(X).$$

If we have a fixed morphism $f : X \to S$ we say that G acts on S-automorphisms of X if the image of $\rho : G \to \mathrm{Aut}(X)$ is contained in the subgroup $\mathrm{Aut}_S(X)$, in otherwords if we have $f \circ \rho(\sigma) = f$ for all $\sigma \in G$.

Example 1.5.2. Let G be a finite group and X a scheme (or more generally any object X in a category with finite coproducts). Then G_X denotes the scheme $\coprod_{\sigma \in G} X_\sigma$, where $X_\sigma = X$ for each σ.

For a fixed $\sigma \in G$ and any $\tau \in G$ consider the composition

$$X_\tau \overset{id_X}{\to} X_{\tau\sigma} \hookrightarrow G_X$$

In this way we get induced an automorphism $\rho(\sigma) : G_X \to G_X$ with inverse $\rho(\sigma^{-1})$. Note that we also have $\rho(\sigma_1\sigma_2) = \rho(\sigma_2) \circ \rho(\sigma_1)$ thus in this way we get a right group action of G on G_X.

Now suppose that $f : X \to S$ is a morphism and we have a group action $\mu : G \to \mathrm{Aut}_S(X)$. For a fixed $\sigma \in G$, the two morphisms $(id_X, \mu(\sigma))$ induce a morphism $X_\sigma = X \to X \times_S X$ and thus we get an S-morphism $\psi_{X/S} : G_X \to X \times_S X$. Moreover if $S' \to S$ is any other S-scheme (or more generally an object in the slice category of S), then letting $X' := S' \times_S X$ (assuming the fibered product exists), with projection $p_X : X' \to X$, we get induced a group action $\mu' : G \to \mathrm{Aut}_{S'}(X')$ where $\mu'(\sigma) : X' \to X'$ is the unique morphisms such that $p_X \circ \mu'(\sigma) = \mu(\sigma) \circ p_X$. Using universal properties it is readily checked that the following diagram where the vertical arrows are the canonical ones is commutative:

$$\begin{array}{ccc} G_{X'} & \xrightarrow{\psi_{X'/S'}} & X' \times_{S'} X' \\ \downarrow & & \downarrow \\ G_X & \xrightarrow{\psi_{X/S}} & X \times_S X \end{array}$$

If the morphism $f : X \to S$ is a finite morphism of schemes, then so is $\psi_{X/S}$.

Definition 1.5.3. Suppose that $\mathcal{C}$ is a category with finite coproducts and G a finite group.

1. For any $X \in \mathcal{C}$ we will call the object $G_X = \coprod_{\sigma \in G} X_\sigma$ [8] constructed in Example 1.5.2 the *object associated to the pair* (X, G).

[7] A right group action is a group homomorphism $\rho : G^{op} \to \mathrm{Aut}(X)$

[8] Some authors denote this $X \times G$.

2. If G acts on S-automorphisms of X, we shall denote the scheme G_X by $G_{X/S}$ and call the morphism $\psi_{X/S} : G_{X/S} \to X \times_S X$ as constructed in Example 1.5.2 the *graph of the pair* $(X, G)/S$.

Definition 1.5.4. Let $\rho : G \to \mathrm{Aut}(X)$ be a group action on an object X in $\mathscr{C}$. A morphism $f : X \to Y$ is G-invariant if

$$f \circ \rho(g) = f$$

for all $g \in G$. A (categorical) group quotient is a morphism

$$\pi : X \to X/G$$

satisfying the following universal property: If $f : X \to Y$ is G-invariant, then there exists a unique morphism

$$\hat{f} : X/G \to Y$$

such that

$$f = \hat{f} \circ \pi.$$

Remark 1.5.5. There are other equivalent ways of defining the group quotient.

1. (Group quotient in terms of corepresentability) The group quotient can also be defined in terms of co-representable functors; For an object X and a group action of G on X, let $(h^X)^G$ be the subfunctor of h^X given by
$$Y \mapsto \{f \in \mathrm{Hom}(X, Y) \mid f \text{ is } G - \text{invariant}\}.$$
The group quotient exists if and only if the functor $(h^X)^G$ is co-representable by an object X/G and in this case the universal element (also called universal morphism) $\pi : X \to X/G$ is the group quotient as in Definition 1.5.4. Indeed if $f \in (h^X)^G(Y)$ then following the proof of the co-Yoneda lemma, there is a unique morphism $\hat{f} \in h^{X/G}(Y)$ such that
$$(h^X)^G(\hat{f})(\pi) = \hat{f} \circ \pi = f.$$
So we see that π is the group quotient. Conversely if $\pi : X \to X/G$ is a categorical group quotient, then we have a natural transformation
$$\eta : h^{X/G} \to (h^X)^G$$
given by
$$(f : (X/G) \to Y) \mapsto f \circ \pi$$
and it is clear from the universal property that η is a natural isomorphism.

2. (Group quotient as a co-equalizer) The group quotient can also be defined in terms of co-equalizers: Suppose $\mathcal{C}$ is a category with finite coproducts and suppose that for a morphism $X \to S$ the fibered product $X \times_S X$ exists. Let G be a group acting on S-automorphisms. If the group quotient X/G exists it is the co-equalizer of the following commutative diagram

$$G_{X/S} \underset{pr_2 \circ \psi_{X/S}}{\overset{pr_1 \circ \psi_{X/S}}{\rightrightarrows}} X$$

Example 1.5.6.

1. For a finite integer d and a vector space we have that $\mathrm{Sym}^d(V) = V^{\otimes d}/\Sigma_d$.

2. The group quotient exists in Sets and it is just the set of orbits under the group action of G.

3. Similarly the set of orbits under the group action of G on a topological space endowed with the finest topology making $X \to X/G$ continous yields the group quotient.

4. For an algebraic field extension $K \subset L$, consider the category of subfields of L containing K where there is a morphism $E \to E'$ if and only if $E' \subset E$. The Galois group $G = \mathrm{Gal}(L/K)$ acts on L canonically and the morphism $L \to L^G$ is the group quotient. Recall that L/K is Galois if and only if $L/G = K$.

5. The previous examples generalizes. Indeed consider the opposite category of commutative rings CRing^{op} and let $\rho : G \to \mathrm{Aut}(B)$ for some ring B. Then the *ring of* G*-invariants*

$$B \to B^G = \{b \in B \mid \rho(g)(b) = b \text{ for all } g \in G\}$$

corresponding to the canonical inclusion $B^G \hookrightarrow B$ is the group quotient.

From this point on when we say that a group G acts on an object X if confusion is not likely to arise we will omit writing the group homomorphism $\rho : G \to \mathrm{Aut}(X)$ and just write $\sigma \in G$ when we really mean $\rho(\sigma) \in \rho(G) \subset \mathrm{Aut}(X)$.

Group quotients of schemes

The group quotient of an affine scheme

In this subsection we will start by showing that if G is a finite group acting on a ring A, then the canonical map $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is the group quotient $\mathrm{Spec}(A)/G$. To do this we will first need to explain some notation and concepts and provide a couple of auxiliary results.

Localization and G-invariants

Suppose that a group G acts on a ring A, and let S be a multiplicative subset of A satisfying $\sigma(S) = S$ for every $\sigma \in G$. Then for each $\sigma \in G$ we get induced an automorphism of $S^{-1}A$ by $a/s \mapsto \sigma(a)/\sigma(s)$, hence we get induced an action of G on $S^{-1}A$.

Lemma 1.5.7. *Let A be a ring, G a group acting on A such that for all $a \in A$ the orbit $\{\sigma(a) \mid \sigma \in G\}$ is finite. Suppose that S a multiplicative subset of A^G. Then the canonical morphism*

$$S^{-1}A^G \to (S^{-1}A)^G$$

is an isomorphism.

Proof. The map is clearly injective. For surjectivity suppose that a/t is G-invariant, so

$$\sigma(a/t) = \sigma(a)/t = a/t$$

for all $\sigma \in G$. Or in other words letting

$$\operatorname{Stab}_G(a) = \{\sigma \in G \ : \ \sigma(a) = a\}$$

denote the stabilizer of a, we have for each $\sigma \in G/\operatorname{Stab}_G(a)$ some $s_\sigma \in S$ such that

$$s_\sigma t\sigma(a) = s_\sigma ta.$$

Since the orbit of a is finite, we have that the group $G/\operatorname{Stab}_G(a)$ is finite hence setting

$$s := \prod_{\sigma \in G/\operatorname{Stab}_G(a)} s_\sigma$$

then $sta \in A^G$ and $sta/t^2s \mapsto a/t$. □

G-invariants of pushforwards of structure sheaves

Suppose that X is a ringed space and G is a group acting on X. For any G-invariant morphism $\pi : X \to Y$ we have that for any open $U \subset Y$, it follows that $\sigma^{-1}(\pi^{-1}(U)) = \pi^{-1}(U)$ for all $\sigma \in G$. The proof of the following lemma is a straightforward verification of the sheaf axioms.

Lemma 1.5.8. *Let X be a ringed space with a group G acting on X. Suppose that $\pi : X \to Y$ is a G-invariant morphism.*

The functor $\pi_*\mathcal{O}_X^G$ given by

$$\pi_*\mathcal{O}_X^G(U) = \{f \in \pi_*\mathcal{O}_X(U) \mid \sigma^\#(\pi^{-1}(U))(f) = f\}$$

is a subsheaf of $\pi_*\mathcal{O}_X$.

The group quotient of an affine scheme exists in the category of schemes

If G acts on the ring A then we get a canonical action of G on $\operatorname{Spec} A$ and clearly the canonical morphism $\pi : \operatorname{Spec} A \to \operatorname{Spec}(A^G)$ is G-invariant.

Proposition 1.5.9. *(Notation as above) The morphism $\pi^{\#} : \mathcal{O}_{\operatorname{Spec}(A^G)} \to \pi_* \mathcal{O}_{\operatorname{Spec} A}$ induces an isomorphism*

$$\mathcal{O}_{\operatorname{Spec}(A^G)} \to \pi_* \mathcal{O}_{\operatorname{Spec} A}^G$$

Proof. It is enough to prove this on the distinguished affine opens, which we have already done in Lemma 1.5.7. $\square$

We are now ready to prove that the quotient of an affine scheme with a finite group always exists.

Proposition 1.5.10 ([Gro71, Exposé V, Prop. 1.1, page 106])**.** *Let A be a ring and let G be a finite group acting on A. Set $X = \operatorname{Spec} A$ and let $Y = \operatorname{Spec} A^G$, and let $\pi : X \to Y$ be the morphism induced by the inclusion $A^G \hookrightarrow A$. The following statements are true:*

(1) *A is integral over A^G, thus π is an integral morphism.*

(2) *The morphism π is surjective, the fibers are the orbits of G in the following sense: If $\mathfrak{q} \in X$, then the fiber over $\mathfrak{p} = \pi(\mathfrak{q}) = \mathfrak{q} \cap A^G$ is the set $\{\sigma(\mathfrak{q}) \mid \sigma \in G\}$. Moreover $\pi : X \to Y$ considered as a map of topological spaces is a quotient map.*

(3) *For $x \in X$, $y = \pi(x)$, $G_x = \operatorname{Stab}_G(x)$ the stabilizer of x, then $k(x)$ is a normal field extension of $k(y)$ and the canonical morphism $G_x \to \operatorname{Gal}(k(x)/k(y))$ is surjective.*

(4) *(Y, π) is the group quotient X/G in the category of ringed spaces (hence also in the category of schemes).*

Proof. **For (1):** Let $x \in A$. The polynomial $P(Y) = \prod_{\sigma \in G}(Y - \sigma(x))$ is a monic polynomial with coefficients in A^G satisfying $P(x) = 0$.

For (2): Surjectivity follows from (1) and the Lying over theorem for integral extensions. Since integral morphisms are closed and closed surjections of topological spaces are quotient maps, it follows that π is a quotient map on the level of topological spaces as claimed. To see what the fibers are first note that if $\mathfrak{q} \in \operatorname{Spec} A$ and if we set $\mathfrak{p} = \mathfrak{q} \cap A^G$, then we have $\mathfrak{p} = \sigma(\mathfrak{q}) \cap A^G$ for any $\sigma \in G$. Now suppose that $\mathfrak{q}, \mathfrak{q}'$ are any two different prime ideals lying over the same prime ideal $\mathfrak{p}$ of A. We will show that there is some $\sigma \in G$ such that $\sigma(\mathfrak{q}) = \mathfrak{q}'$. Since A is integral over A^G we have by [AM69, Corollary 5.9] that there cannot be any inclusion relations between $\mathfrak{q}$ and $\mathfrak{q}'$. Further by prime avoidance we will be done if we can show that $\mathfrak{q}' \subset \cup_{\sigma \in G} \sigma(\mathfrak{q})$. Suppose that this is not the case, then we can find some $x \in \mathfrak{q}'$ such that $x' \notin \sigma(\mathfrak{q})$ for all $\sigma \in G$. Set $y = \prod_{\sigma \in G} \sigma(x)$. We have that $y \in A^G \cap \mathfrak{q}'$ and thus also $y \in \mathfrak{q} \cap A^G$, so $y \in \mathfrak{q}$ thus we must have $\sigma(x) \in \mathfrak{q}$ for some $\sigma \in G$, but then $x \in \sigma^{-1}(\mathfrak{q})$ which gives the desired contradiction.

For (3): See [Bou64, Ch 5, Sec. 2, n.2, Theorem 2] or [Stacks, Tag 0BRJ].

For (4): Suppose that $\tau : X \to Z$ is a G-invariant morphism of ringed spaces. Since τ is G-invariant, we have that $\tau(\sigma(x)) = x$ for all $\sigma \in G$ and since by (2) the orbit $\{\sigma(x) \mid \sigma \in G\}$ is the fiber of $\pi(x)$ for each $x \in X$ and π is a quotient map, there exists a unique map of topological spaces $\hat{\tau} : Y \to Z$ such that $\tau = \hat{\tau} \circ \pi$ as maps of topological spaces. Now to construct $\hat{\tau}^\# : \mathcal{O}_Z \to \hat{\tau}_* \mathcal{O}_Y$ note that since τ is G-invariant the map $\tau^\#$ factors through the inclusion $(\tau_* \mathcal{O}_X)^G \hookrightarrow \tau_* \mathcal{O}_X$ and since (by Proposition 1.5.9) we have

$$(\tau_* \mathcal{O}_X)^G = (\hat{\tau} \circ \pi)_* \mathcal{O}_X^G = \hat{\tau}_* (\pi_* \mathcal{O}_X^G) \cong \hat{\tau}_* \mathcal{O}_Y,$$

we get our morphism $\hat{\tau}^\#$. Uniqueness is clear. □

Example 1.5.11. (The real affine line) Let $A = \mathbb{C}[x]$ and $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$. The group G acts on A by conjugating constants. Then $A^G = \mathbb{R}[x]$. Hence we have that $\pi : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ is the group quotient. This gives us a new way of computing $\mathbb{A}^1_{\mathbb{R}}$. Indeed we know that π is surjective, and we can compute its image. If $a, b \in \mathbb{R}$ are any two elements of $\mathbb{R}$ with $b \neq 0$. Then a point of the form $[(x-a)] \in \mathbb{A}^1_{\mathbb{C}}$ is mapped to $[(x-a)] \in \mathbb{A}^1_{\mathbb{R}}$, and as $(x-(a+ib))(x-(a-ib)) = x^2 - 2ax + (a^2+b^2) \in \mathbb{R}[x]$ is irreducible we see that both $[(x-(a+ib))] \in \mathbb{A}^1_{\mathbb{C}}$ and $[(x-(a-ib))] \in \mathbb{A}^1_{\mathbb{C}}$ are mapped to $[(x^2 - 2ax + (a^2+b^2))]$. Thus we see that the affine real line is obtained from the affine complex line where complex conjugates have been glued together.

Note that we have also proved that any polynomial with real coefficients can be factored as a product of real polynomials each of degree at most two, which is a fact one might have already proved in a first course on calculus.

Example 1.5.12. (The affine cone as a group quotient) Consider the $\mathbb{Z}/2\mathbb{Z}$-action on the ring $k[x,y]$ given by $x \mapsto -x, y \mapsto -y$. This induces a group action on $\mathbb{A}^2_k$. It is easy to see that the subring of $\mathbb{Z}/2\mathbb{Z}$-invariants is the subring generated by xy, x^2, y^2. Furthermore we have a surjective morphism

$$k[u,v,w] \to k[x,y]^{\mathbb{Z}/2\mathbb{Z}}$$

given by

$$u \mapsto x^2, v \mapsto y^2, w \mapsto xy$$

It is clear that the ideal $(uv - w^2)$ is contained in the kernel of this map, moreover since

$$(0) \subset (xy, x^2) \subset (xy, x^2, y^2)$$

is a chain of prime ideals in $k[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ we have that $k[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ is at least two dimensional, and since $(uv-w^2)$ is prime we then conclude that we have an isomorphism

$$k[u,v,w]/(uv-w^2) \cong k[x,y]^{\mathbb{Z}/2\mathbb{Z}}.$$

Conditions under which the group quotient of schemes exist

Proposition 1.5.13 ([Gro71, Exposé V, Prop. 1.3, page 107])**.** *Let X be a scheme and G a finite group of automorphisms of X. Suppose that $p : X \to Y$ is an affine G-invariant morphism such that*

$$\mathcal{O}_Y \to p_*\mathcal{O}_X^G$$

is an isomorphism. Then the conclusions of (1),(2),(3) and (4) of Proposition 1.5.10 are still valid.

Proof. Since p is G-invariant it follows that $\sigma^{-1}(p^{-1}(U)) = p^{-1}(U)$ for every $\sigma \in G$ and open $U \subset Y$. Further since p is affine, it follows that $p^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ for some ring A, and from the assumption it then immediately follows that $B = A^G$. From the aforementioned remarks, assertions (1), (2) and (3) follow because they can all be checked on open affine covers. Finally (4) follows by applying (2) to see that p is a quotient map on the level of topological spaces and use that $\mathcal{O}_Y = p_*\mathcal{O}_X^G$. □

Corollary 1.5.14. *Under the conditions of Proposition 1.5.13, for every open U of Y, U is the quotient of $p^{-1}(U)$ of G.*

Proof. The morphism $p^{-1}(U) \to U$ induced by p satisfies the same hypothesis as p. □

For the rest of this section, we shall always assume that X is a scheme over Z and the group G acts on Z-automorphisms of X.

Corollary 1.5.15. *(Under the hypothesis of Proposition 1.5.13). The morphism $X \to Z$ is affine (respectively separated) if and only if the induced morphism $Y \to Z$ is. If X is of finite type over Z, then the morphism $p : X \to Y$ is finite. Moreover if X is of finite type over Z and Z is locally Noetherian, then $Y \to Z$ is of finite type.*

Proof. A clear proof can be found in [Gro71, Exposé V, Corollaire 1.5. , pages 107-108]. □

Definition 1.5.16 ([Gro71, Exposé V, Définition 1.7.,page 109])**.** Suppose X is a scheme and G a finite group acting (on the right) on X. We say that G acts admissibly if there exists a morphism $p : X \to Y$ with the properties of the morphism p from Proposition 1.5.13.

The following result is part of [Gro71, Exp. V, Prop. 1.8].

Lemma 1.5.17. *Let X be a scheme and suppose G is a finite group acting on X. The following are equivalent*

(1) *The scheme X is a union of affine open subsets which are invariant under G. That is X can be covered by open affine subsets U satisfying $\sigma(U) = U$ (as sets) for all $\sigma \in G$.*

(2) *For every $x \in X$ the orbit $\{\sigma(x) \mid \sigma \in G\}$ is contained in an open affine subset of X.*

Proof. Clearly (1) implies (2). To see that (2) implies (1) it is enough to show that an arbitrary orbit T is contained in an affine open subscheme of X. For this purpose note that by assumption we can find some affine open subscheme V of X such that $T \subset V$. Let W be the open set

$$W := \cap_{\sigma \in G} \sigma(V).$$

For an arbitrary $\sigma \in G$ note that since σ is an automorphism of X and T is an orbit we have

$$T = \sigma(T) \subset \sigma(V); \tag{1.5.1}$$

$$\sigma(W) = \cap_{\sigma' \in G} \sigma(\sigma'(V)) = \cap_{\sigma'' \in G} \sigma''(V). \tag{1.5.2}$$

Hence W contains T, and is G-invariant. If X is separated then W is also affine as desired and we claim that we can in fact assume that X is separated. Indeed since W is always separated we only need to show that there is an affine open subset W' of W containing T. To find such a subset W' simply observe that W is an open subset of an affine scheme and apply prime avoidance. $\square$

Proposition 1.5.18 ([Gro71, Exposé V, Proposition 1.8, page 109])**.** *Let X be a scheme and G a finite group acting on X. The group G acts admissibly if and only if either of the equivalent conditions of Lemma 1.5.17 are satisfied.*

Proof. The condition is necessary because if $p : X \to Y$ satisfies the hypothesis of Proposition 1.5.13, then for $x \in X$ we can find some affine open subset $V \subset Y$ containing $f(x)$ and we have that the orbit of x is the fiber of $f(x)$ which is contained in $p^{-1}(V)$ which is affine by assumption on p.

For sufficiency, suppose $X = \cup X_i$ is a cover of open affines with each X_i being invariant under G. Then $Y_i = X_i/G$ exists, and we have morphisms $\pi : X_i \to Y_i$ satisfying the properties of Proposition 1.5.13. Since p_i is quotient map and by assumption on the cover X_i we have that

$$p_i^{-1}(p_i(X_i \cap X_j)) = X_i \cap X_j,$$

it follows that $Y_{ij} := p_i(X_i \cap X_j)$ is an open subset of Y_i. Moreover we have by Corollary 1.5.14 that Y_{ij} is the group quotient of $X_i \cap X_j$, but so is Y_{ji} so we get induced isomorphisms $Y_{ij} \cong Y_{ji}$. Now it follows from the universal property of group quotients that these isomorphisms satisfy the cocycle condition. Thus we can glue the Y_i along the Y_{ij} and get a scheme Y. Moreover by construction we can glue the p_i to get a morphism $p : X \to Y$. It is clear that p is an affine morphism. It is also clear that p is G-invariant since it is on a cover of X and moreover the morphism

$$p^{\#} : \mathcal{O}_Y \to p_* \mathcal{O}_X^G$$

is an isomorphism because it is on a cover of Y.

$\square$

Remark 1.5.19. Note that the proof of Proposition 1.5.18 shows that if G is a finite group acting admissibly on X and U is any G-invariant open subset of X then the induced map $U/G|_U \to X/G$ is an open embedding and we have $p^{-1}(U/G|_U) = U$.

Corollary 1.5.20. *If the finite group G acts admissibly on a scheme X, then so does any subgroup H of G (thus X/H exists).*

Corollary 1.5.21. *Suppose $X \to Z$ is an affine morphism and G acts on the Z-automorphisms. Then G acts admissibly on X. If $\mathcal{A}$ is the quasi-coherent sheaf of algebras on Z defining X, then the quotient Y is defined by the sheaf of $\mathcal{O}_Z$-algebras $\mathcal{A}^G$ of G-invariants of $\mathcal{A}$.*

Proof. If $X \to Z$ is affine then it is clear that the conditions of Lemma 1.5.17 are satisfied, thus by Proposition 1.5.18 G acts admissibly on X. We can describe the group quotient $X \to Y = X/G$ more explicitly as follows. If $\beta : X = \underline{\mathrm{Spec}}_Z(\mathcal{A}) \to Z$, then we have $\beta_* \mathcal{O}_X \cong \mathcal{A}$ and for any $\sigma \in G$, the morphism

$$\beta_* \sigma^{\#} : \beta_* \mathcal{O}_X \to \beta_*(\sigma_* \mathcal{O}_X) = \beta_* \mathcal{O}_X$$

canonically induces a morphism

$$\sigma_{\mathcal{A}}^{\#} : \mathcal{A} \to \mathcal{A}$$

We define the sheaf of $\mathcal{O}_Z$-algebras $\mathcal{A}^G$ as follows

$$\mathcal{A}^G(U) := \{f \in \mathcal{A}(U) \mid \sigma_{\mathcal{A}}^{\#}(U)(f) = f \text{ for all } \sigma \in G\} = \mathrm{Ker}(\prod_{\sigma \in G} (\sigma_{\mathcal{A}}^{\#} - id_{\mathcal{A}})).$$

The sheaf $\mathcal{A}^G$ is a subsheaf of $\mathcal{A}$ and induces a morphism $X \to \underline{\mathrm{Spec}}_Z(\mathcal{A}^G)$ which by construction is the group quotient X/G. $\square$

Lemma 1.5.22. *Let X, Y be schemes and $\{X_i\}$ be an open covering of X. Suppose that we have morphisms $f_i : X_i \to Y$ and for each pair i, j we have an open cover $\{U_{i,j,k}\}_{k \in \mathcal{K}}$ (where $\mathcal{K}$ depends on the pair i, j) of $X_i \cap X_j$ such that the restrictions $f_i|_{U_{i,j,k}} = f_j|_{U_{i,j,k}}$ for all i, j, k. Then the f_i glue to a morphism $f : X \to Y$.*

Proof. This is part of the fact that representable functors are sheaves on the big Zariski site. □

Example 1.5.23. Suppose X is an integral scheme and L is a finite field extension of $k(X)$ the field of functions of X. Let $G = \mathrm{Aut}(L/k(X))$ be the Galois group of the extension $L/k(X)$. Then we get a right action of G $\rho : G^{op} \to \mathrm{Aut}_X(\tilde{X})$ where $\tilde{X} \to X$ denotes the normalization of X in L. Indeed for any affine open $U \cong \operatorname{Spec} A$ of X, then $\nu^{-1}(U) \cong \operatorname{Spec} B$ where B is the integral closure of A in L. For any $\sigma \in G$ we have $\sigma(B) = B$, thus we get induced a morphism of A-algebras $\operatorname{Spec}(\sigma) : \operatorname{Spec} B \to \operatorname{Spec} B$. It is not hard to check that these morphisms satisfy the conditions of Lemma 1.5.22 and hence glue to an X-automorphism $\rho(\sigma) : \tilde{X} \to \tilde{X}$.

It is easy to see that G acts admissibly on $\tilde{X}$. Note also that if the extension $L/k(X)$ is Galois and X is a normal connected scheme, then the normalization morphism $\nu : \tilde{X} \to X$ is the group quotient.

Example 1.5.24. Suppose that $X = \mathrm{Proj}(S) \setminus V_+(I)$ is a quasi-projective scheme and let

$$T = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

be a finite set in X. Then by assumption we have that $I \not\subseteq \mathfrak{p}_i$ for every $\mathfrak{p}_i$ in T. Thus by graded prime avoidance we have some homogeneous $f \in I$ such that $f \notin \mathfrak{p}_i$ for all i, then we have $D_+(f) \cap V_+(I) = \emptyset$ thus $D_+(f) \subset X$ and $T \subset D_+(f)$. Hence every finite set in a quasiprojective A-scheme is contained in some affine open subscheme. From this it follows that if G is a finite group acting on a quasi projective scheme, then the group acts admissibly and so the group quotient $X \to X/G$ exists.

More generally we have that

Proposition 1.5.25. *Suppose that $f : X \to S$ is a quasi-compact morphism of schemes. Suppose that T is a finite subset of X and $f(T)$ is a one point set (i.e, all elements of T are mapped to the same point in S). If X has an f-ample invertible sheaf $\mathcal{L}$, then T is contained in an open affine subset of X. In particular if G is a finite group acting by S-automorphisms on X, then G acts admissibly.*

Proof. Recall that if $\mathcal{L}$ is f-ample, then by [GD61, Prop. 4.6.3] or [Stacks, Tag 01VJ] there exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and an open embedding $X \to \underline{\mathrm{Proj}}_S \mathcal{A}$ such that

$$\begin{tikzcd} X \arrow[rr, hook, "\circ"] \arrow[dr, "f"'] & & \underline{\mathrm{Proj}}_S \mathcal{A} \arrow[dl] \\ & S & \end{tikzcd}$$

commutes.

Now if T is mapped to the point $s \in S$ then for any open neighborhood U of s we have that $T \subset f^{-1}(U)$ is an open subset of X isomorphic to an open subset of $\mathrm{Proj}(B)$ for some ring B and by Example 1.5.24 we get that T is contained in an affine open subset of X. □

In the previous Proposition 1.5.25 we saw an example of a scheme with the property that finite sets over a given point are contained in an affine subset. We now follow [Ryd08c] and give this a name:

Definition 1.5.26. Let X/S be a scheme over S. We say that X/S is AF if the following condition is satisfied:

(AF) Every finite set of points $x_1, \dots, x_n$ over the same point $s \in S$ is contained in an affine open subset of X.

Remark 1.5.27. If a finite group acts by S-automorphisms on an *AF* scheme X/S then G acts admissibly on X.

Remark 1.5.28. Although this notion may not have been given a name until relatively recently, it has frequently been used as a natural setting to state and prove problems for a rather long time. An interesting example of this is Chevalley's conjecture stating that a nonsingular proper variety is projective if and only if it is AF. The conjecture was settled affirmatively by Kleiman in [Kle66].

Remark 3.1.3 of [Ryd08c] mentions that AF morphisms are necessarily separated and stable under base change. We will now list a few other properties of AF schemes.

Lemma 1.5.29. *We have the following properties of* AF *morphisms.*

(1) *If X/S and Y/S are* AF *then so is $X \times_S Y \to S$.*

(2) *Let Λ be a set and suppose that we have S-morphisms $X_\lambda \to S$ for every $\lambda \in \Lambda$ such that X_λ/S is* AF *for every $\lambda \in \Lambda$. Then the induced S-scheme*

$$X := \coprod_{\lambda \in \Lambda} X_\lambda \to S$$

is AF.

(3) *If X/S is* AF *and G is a finite group acting on S-automorphisms of X then the quotient $X/G \to S$ which exists is also* AF.

Proof. **For Item (1):** note that if $z_1, \ldots, z_n$ are points on $Y \times_S X$ lying over the same point $s \in S$ then it is clear that the images of these points in X and Y respectively lie over the same point $s \in S$. By assumption we have open affine open subsets $V \subset Y$ and $U \subset X$ containing these points. Furthermore from prime avoidance it is clear that if W is an open subset of S then $X \times_S W/W$ and $Y \times_S W/W$ are AF (this is a special case of the AF property being stable under base change ([Ryd08c, Rmk. 3.1.3])). Thus we can reduce to the case where the scheme S is affine in which case the scheme $U \times_S V$ is an affine open subscheme of $Y \times_S X$ containing the points $z_1, \ldots, z_n$.

For Item (2): It is clearly enough to prove the case of a finite coproduct. Moreover by induction we reduce to the case $X = Y \coprod Z$. Let now $y_1, \ldots, y_n \in Y$ and $z_1, \ldots, z_m \in Z$ be points of X lying over the same point $s \in S$. Since Y and Z are AF we can find affine opens $U \subset Y$ containing all the y_i and $V \subset Z$ containing all the z_j. The induced map $U \coprod V \to X = Y \coprod Z$ is clearly an open embedding and since a finite coproduct of affine schemes remains affine we are done.

For Item (3): See [Har16, Lemma 3.1.12.(e)]. □

Group quotients are well behaved with respect to flat base change:

Proposition 1.5.30 ([Gro71, Exposé V, Proposition 1.9, page 109])**.** *Suppose G acts admissibly on X, and $X/G = Y$ a scheme over Z. Consider a base change $Z' \to Z$, and set $X' = X \times_Z Z'$, $Y' = Y \times_Z Z'$. Then we get induced an action of G on X' ($\sigma \mapsto \sigma \times id_{Z'}$ for $\sigma \in G$). The morphism $p' : X' \to Y'$ induced by $p : X \to Y$ is G-invariant. Further if Z' is flat over Z, then p' satisfies the hypothesis of Proposition 1.5.13, i.e. $\mathcal{O}_{Y'} \to p'_*\mathcal{O}_{X'}^G$ is an isomorphism. Thus G acts admissibly on X', and*

$$(X/G) \underset{Z}{\times} Z' \cong (X \times_Z Z')/G.$$

Proof. A clear proof is written in [Gro71]. Flatness is necessary for the proposition to be true. □

Corollary 1.5.31. *Let G, H be two finite groups acting admissibly on S-automorphisms of X and Y respectively. Then we get induced an admissible action of $G \times H$ on $X \times_S Y$. Moreover if both the quotients X/G and Y/H are flat over S then the canonical map*

$$X \underset{S}{\times} Y \to (X/G) \underset{S}{\times} (Y/H) \tag{1.5.3}$$

is the quotient of $X \times_S Y$ with the action of $G \times H$.

1.6 Symmetric powers

We start by defining the notion of symmetric powers for sets which will make it easier to understand the case of schemes coming up next.

Symmetric powers of sets

Suppose that X is a set and let d be a positive integer. Then for any permutation of d elements $\sigma \in \Sigma_d$ we get an automorphism $\rho(\sigma)$ of $X^{\times d}$ given by $(x_1, \ldots, x_d) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$. Note that since $\rho(id) = id_{X^{\times d}}$ and for any two permutations σ_1, σ_2 we have $\rho(\sigma_1 \circ \sigma_2) = \rho(\sigma_1) \circ \rho(\sigma_2)$, we have a group homomorphism $\rho : \Sigma_d \to \mathrm{Aut}(X^{\times d})$.

We say that a map $f : X^{\times d} \to S$ is Σ_d-invariant if

$$f \circ \rho(\sigma) = f$$

for all $\sigma \in \Sigma_d$.

Now we can define an equivalence relation on $X^{\times d}$ as follows: We say that $(x_1, \ldots, x_d) \in X^{\times d}$ is equivalent to $(y_1, \ldots, y_d) \in X^{\times d}$ if and only if there is some $\sigma \in \Sigma_d$ such that $\rho(\sigma)(x_1, \ldots, x_d) = (y_1, \ldots, y_d)$. We let $\mathrm{Sym}^d(X)$ be the quotient of $X^{\times d}$ with the aforementioned equivalence relation. We then have a canonical map $\pi : X^{\times d} \to \mathrm{Sym}^d(X)$ and for any Σ_d-invariant map $f : X^{\times d} \to S$ we have a unique map $\hat{f} : \mathrm{Sym}^d(X) \to S$ satisfying

$$f = \hat{f} \circ \pi$$

We call the set $\mathrm{Sym}^d(X)$ the *d'th symmetric power of* X.

Remark 1.6.1. Note that this construction does not completely resemble the vector space case, as the tensor product is not a product in the category of vector spaces. The way to see how the two notions are analogous is through symmetric monoidal categories, as vector spaces with tensor products and sets with products are both examples of such categories.

Symmetric tensors of rings

Before introducing symmetric powers of schemes we will first briefly recapitulate the ring theoretic version. We will only briefly recall the basic definitions and properties, such as flatness and an explicit description of the generators of the symmetric tensors without giving any proofs. For a more thorough treatment of this topic we refer the reader to Chapter 2 of [Har16].

Definition 1.6.2. Let n be a non-negative integer and M a module over a ring A. The *n-fold tensor product* of M over A will be denoted as

$$(M/A)^{\otimes n} := M^{\otimes_A^n} \tag{1.6.1}$$

The n-fold tensor product has a natural action of the symmetric group S_n by permuting the n tensor factors. The corresponding module of invariants

$$\mathrm{S}_n(M/A) := \left((M/A)^{\otimes n}\right)^{\Sigma_n} \tag{1.6.2}$$

is called the n-th symmetric tensors of M over A.

Remark 1.6.3. If B is an A-algebra, then so are $(B/A)^{\otimes n}$ and $\mathrm{S}_n(B/A)$.

Lemma 1.6.4. *Let A be a ring and let n be a non-negative integer. Suppose that M is a flat A-module. Then the A-module $\mathrm{S}_n(M/A)$ is also flat.*

Proof. See [Har16, Theorem 2.1.14]. □

Definition 1.6.5. Let B be an algebra over a ring A and let n be a non-negative integer.

1. For every $k \in \{1, 2, \ldots, n\}$ the *k-th formal n-tensor conjugate of* $b \in B$ over A is the pure tensor

$$\iota_k(b) := \iota_k^n(b) := 1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes \ldots \otimes 1 \in (B/A)^{\otimes n} \tag{1.6.3}$$

   with b at position k and 1 at all other places.

2. For every $k \in \{0, 1, \ldots, n\}$ the k-th *elementary symmetric n-tensor* $\rho_k(b) := \rho_k^n(b)$ of $b \in B$ over A is the coefficient of x^{n-k} in the polynomial

$$(x + b)^{\otimes n} = \prod_{i=1}^{n} (x + \iota_i(b)) \in (B[x]/A[x])^{\otimes n} \cong (B/A)^{\otimes n}[x] \tag{1.6.4}$$

   It is clearly an element of $\mathrm{S}_n(B/A)$.

3. An element of $(B/A)^{\otimes n}$ is called an *elementary symmetric n-tensor* if it is the k-th elementary symmetric n-tensor of some $b \in B$ for some k. If it is clear from the context, we will omit n.

Remark 1.6.6. If σ_k is the k-th elementary symmetric polynomial in n-variables we have

$$\rho_k^n(b) = \sigma_k(\iota_1(b), \ldots, \iota_n(b)) = \sum_{1 \leq j_1 < j_2 < \ldots < j_k < n} \iota_{j_1}(b) \cdot \ldots \cdot \iota_{j_k}^n(b). \tag{1.6.5}$$

Theorem 1.6.7. *Let A be a ring and n a non-negative integer. Let B be a flat A-algebra which is, as an A-module, generated by a subset $E \subset B$.*

Then the ring $\mathrm{S}_n(B/A)$ of invariants is generated as an A-algebra by the elementary symmetric n-tensors (Definition 1.6.5) of the $e \in E$.

Proof. See [Har16, Theorem 2.1.18] or [Vac06, Sec. 4.3, Prop. 3]. □

Observation 1.6.8. Let $n_1, \ldots, n_r$ be positive integers and set $N = \sum_{i=1}^{r} n_i$. Suppose that A is a ring and B is an A-algebra and consider the isomorphism

$$\gamma^{((n_i))} : (B/A)^{\otimes N} \to \bigotimes_{i=1}^{r} (B/A)^{\otimes n_i} \tag{1.6.6}$$

given by

$$(b_{1,1} \otimes \ldots \otimes b_{1,n_1} \otimes b_{2,1} \otimes \ldots \otimes b_{2,n_2} \otimes \ldots \otimes b_{r,n_r}) \mapsto \bigotimes_{i=1}^{r} (\otimes_{j=1}^{n_i} b_{i,j}).$$

We can give an explicit description of the image of an elementary symmetric N-tensor under this map:

For $1 \leq k \leq N$ let D_k be the set of of r-tuples $(d_1, \ldots, d_r)$ satisfying $\sum_{i=1}^{r} d_i = k$ and $d_i \leq n_i$ for all i. Let $b \in B$ and consider the sum

$$\sum_{(d_1,\ldots,d_r)\in D_k} \rho_{d_1}^{n_1}(b) \otimes \rho_{d_2}^{n_2}(b) \otimes \ldots \otimes \rho_{d_r}^{n_r}(b) \in \bigotimes_{i=1}^{r} (B/A)^{\otimes n_i} \tag{1.6.7}$$

We claim that this is the image of $\rho_k^N(b)$. Indeed this can be deduced easily by using the description of elementary symmetric n-tensors in terms of elementary symmetric functions (1.6.5) and the combinatorial observation that we have the equality

$$\sum_{(d_1,\ldots,d_r)\in D_k} \prod_{i=1}^{r} \binom{n_i}{d_i} = \binom{N}{k}. \tag{1.6.8}$$

From this observation and Theorem 1.6.7 we see in particular that if B is a flat A algebra then the restriction of $\gamma^{((n_i))}$ to the sub A-algebra $\mathrm{S}_N(B/A)$ factors through $\bigotimes_{i=1}^{r} \mathrm{S}_{n_i}(B/A)$.

Symmetric powers of schemes

The action of Σ_d on $(X/S)^d$

Let $X \to S$ be an S-scheme. We will start by explaining how Σ_d acts on $(X/S)^d := X \times_S X \times_S \ldots \times_S X$. In the previous section we explicitly described the maps $\rho(\sigma)$ and hence how Σ_d was acting on the d'th self product of a set. We recall from the aforementioned example that we have $pr_i \circ \rho(\sigma) = pr_{\sigma(i)}$ and thus the maps $q_i : X^{\times d} \to X$ given by $q_i := pr_{\sigma(i)}$ induce the maps $\rho(\sigma)$. Thus we can do the same thing for schemes as well, indeed define the maps $q_i : (X/S)^d \to X$ by $q_i := pr_{\sigma(i)}$. Then by the universal property of (fibered) products we get induced a map $\rho(\sigma) : (X/S)^d \to (X/S)^d$ satisfying

$$pr_i \circ \rho(\sigma) = pr_{\sigma(i)}$$

We claim that $\rho(\sigma)$ is an automorphism with inverse $\rho(\sigma^{-1})$. Indeed we have

$$pr_i \circ \rho(\sigma^{-1}) \circ \rho(\sigma) = pr_{\sigma^{-1}(i)} \circ \rho(\sigma) = pr_{\sigma(\sigma^{-1}(i))} = pr_i$$

for all i, thus by universal property of fibered products, we must have that $\rho(\sigma^{-1}) \circ \rho(\sigma) = id$.

Symmetric powers of schemes: Definition and existence hypothesis

Definition 1.6.9. If the group quotient $(X/S)^d/\Sigma_d$ (where Σ_d acts as described in the preceeding paragraph) exists, then we define the d'th *symmetric power* of X to be the quotient

$$\mathrm{Sym}^d(X/S) := (X/S)^d/\Sigma_d.$$

Example 1.6.10. Suppose that $f : X \to S$ is an affine morphism, with $\mathcal{A}$ a sheaf of $\mathcal{O}_S$-algebras and $X = \underline{\mathrm{Spec}}_S(\mathcal{A})$. We have that $(X/S)^d = \underline{\mathrm{Spec}}_S(\mathcal{A}^{\otimes_{\mathcal{O}_S} d})$ and by Corollary 1.5.21 we have that

$$\mathrm{Sym}^d(X/S) = \underline{\mathrm{Spec}}_S((\mathcal{A}^{\otimes_{\mathcal{O}_S} d})^{\Sigma_d}).$$

Proposition 1.6.11. *Let d be a positive integer. If X/S is* AF*, then* $\mathrm{Sym}^d(X/S)$ *exists.*

Proof. By Lemma 1.5.29 we have that $(X/S)^d$ is AF, and the group Σ_d gives a (right) group action on $(X/S)^d$ by S-automorphisms, thus by Proposition 1.5.18 Σ_d acts admissibly on $(X/S)^d$. □

Corollary 1.6.12. *Let $f : X \to S$ be a quasi-compact morphism of schemes. If X has an f-ample invertible sheaf, then* $\mathrm{Sym}^d(X/S)$ *exists. In particular symmetric powers of quasi-projective schemes exist.*

Proof. By Proposition 1.5.25, the scheme X/S is AF. □

The following Lemma is essentially [Ryd08c, Remark (3.1.4)].

Lemma 1.6.13. *Let $f : X \to S$ be an* AF *scheme. Let $\{S_\alpha\}_\alpha$ be an open affine cover of S. There exists an open affine cover $\{U_{\alpha,\beta}\}_{\alpha,\beta}$ of X such that*

$$f(U_{\alpha,\beta}) \subset S_\alpha$$

for all α and any subset of d-points of X lying over the same point $s \in S_\alpha$ is contained in some $U_{\alpha,\beta}$. Moreover any such cover $\{U_{\alpha,\beta}\}_{\alpha,\beta}$ has the property that

$$\{((U_{\alpha,\beta})/S_\alpha)^d\}_{\alpha,\beta}$$

is an open cover of $(X/S)^d$. Thus

$$\{Sym^d(U_{\alpha,\beta}/S_\alpha)\}_{\alpha,\beta}$$

is an open affine cover of $\mathrm{Sym}^d(X/S)$*.*

Proof. It is clear from the AF property that we can construct an open affine cover $\{U_{\alpha,\beta}\}_{\alpha,\beta}$ of X with $f(U_{\alpha,\beta}) \subset S_\alpha$ such that any d points lying over the same $s \in S_\alpha$ is contained in some $U_{\alpha,\beta}$.

Now if $x' \in (X/S)^d$ then $pr_1(x'), \ldots, pr_d(x')$ are d points of X lying over the same point s which is contained in some S_α, then we can find some $U_{\alpha,\beta}$ containing $pr_1(x'), \ldots, pr_d(x')$. We have that

$$U_{\alpha,\beta} \underset{S_\alpha}{\times} U_{\alpha,\beta} \underset{S_\alpha}{\times} \ldots \underset{S_\alpha}{\times} U_{\alpha,\beta} = (U_{\alpha,\beta}/S)^d$$

is isomorphic to

$$pr_1^{-1}(U_{\alpha,\beta}) \cap \ldots \cap pr_d^{-1}(U_{\alpha,\beta})$$

which contains x', hence the $(U_{\alpha,\beta}/S)^d$ give an open cover of $(X/S)^d$.

The final assertion follows from noting that the affine cover $(U_{\alpha,\beta}/S_\alpha)^d$ is Σ_d invariant and the construction of $\mathrm{Sym}^d(X/S)$ given in the proof of Proposition 1.5.18. $\square$

Lemma 1.6.14. *Let $X \to S$ be a flat morphism of schemes and suppose that X/S is AF. Then for any non-negative integer n the scheme $\mathrm{Sym}^n(X/S)$ is flat over S.*

Proof. By Lemma 1.6.13 we reduce to the affine case which follows from Lemma 1.6.4. $\square$

Monoidal structure on infinite symmetric powers

In this subsection we work exclusively in the category of S-schemes hence by $\prod_i X_i$ we will mean the product in this category.

Convention 1.6.15. For non-negative integers m, n, r. We let Σ_m denote the symmetric group of bijections $[m] \to [m]$. We will fix the following convention to understand certain permutation groups as subgroups of a symmetric groups.

$\Sigma_m \times \Sigma_n$ will be identified with the subgroup of Σ_{m+n} that leaves the subset $[m]$ and $[m+n] \setminus [m]$ of $[m+n]$ invariant. For non-negative integers $n_1, \ldots, n_r$ this extends inductively to an embedding

$$\Sigma_{n_1} \times \ldots \times \Sigma_{n_r} \hookrightarrow \Sigma_{n_1+\ldots+n_r}. \tag{1.6.9}$$

Lemma 1.6.16. *Let $X \to S$ be a flat AF scheme over S. For positive integers $n_1, \ldots, n_r$ set $N = \sum_{i=1}^r n_i$. Recalling the identifications of Convention 1.6.15 we get induced:*

1. *An isomorphism*

$$\prod_{i=1}^r \mathrm{Sym}^{n_i}(X/S) \cong \left(\prod_{i=1}^r (X/S)^{n_i}\right) / \left(\prod_{i=1}^r \Sigma_i\right) \tag{1.6.10}$$

2. *A unique morphism*

$$\sigma^{(n_i)} : \prod \mathrm{Sym}^{n_i}(X/S) \to \mathrm{Sym}^N(X/S)$$

making the following diagram commutative

$$\begin{array}{ccc} \prod_{i=1}^r (X/S)^{n_i} & \xrightarrow{\cong} & (X/S)^N \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \mathrm{Sym}^{n_i}(X/S) & \xrightarrow{\sigma^{(n_i)}} & \mathrm{Sym}^N(X/S) \end{array} \tag{1.6.11}$$

Proof. **For Item 1:** Lemma 1.6.14 enables the use of Corollary 1.5.31. By induction we conclude.

For Item 2: Since the composition of the rightmost vertical morphism with the top horizontal arrow is clearly $\prod_{i=1}^r \Sigma_{n_i}$ invariant, the existence of the desired morphism $\sigma^{(n_i)}$ follows from Item 1. □

For non-negative integers m, n we will sometimes refer to the map $\sigma^{m,n}$ from Lemma 1.6.16 as *addition*.

Remark 1.6.17. In the notation of Lemma 1.6.16 if $n_1 = n_2 = \ldots = n_r = 1$ and we identify $\mathrm{Sym}^1(X/S)$ with X/S then the addition map

$$\sigma^{(n_i)} : \prod \mathrm{Sym}^1(X/S) = (X/S)^N \to \mathrm{Sym}^N(X/S)$$

is exactly the quotient map $(X/S)^N \to \mathrm{Sym}^N(X/S)$.

Lemma 1.6.18. *For non-negative integers k, m, n we have:*

1. *Commutativity of addition:*

$$\begin{array}{ccc} \mathrm{Sym}^m(X/S) \times \mathrm{Sym}^n(X/S) & \xrightarrow{\sigma^{m,n}} & \mathrm{Sym}^{m+n}(X/S) \\ \downarrow{\scriptstyle swap} & & \| \\ \mathrm{Sym}^n(X/S) \times \mathrm{Sym}^m(X/S) & \xrightarrow{\sigma^{n,m}} & \mathrm{Sym}^{m+n}(X/S). \end{array} \tag{1.6.12}$$

2. *Associativity of addition:*

$$\begin{array}{ccc} \mathrm{Sym}^k(X/S) \times \mathrm{Sym}^m(X/S) \times \mathrm{Sym}^n(X/S) & \xrightarrow{\mathrm{Sym}^k(X/S) \times \sigma^{m,n}} & \mathrm{Sym}^k(X/S) \times \mathrm{Sym}^{m+n}(X/S) \\ \downarrow{\scriptstyle \sigma^{k,m} \times \mathrm{Sym}^n(X/S)} & & \downarrow{\scriptstyle \sigma^{k,m+n}} \\ \mathrm{Sym}^{k+m}(X/S) \times \mathrm{Sym}^n(X/S) & \xrightarrow{\sigma^{k+m,n}} & \mathrm{Sym}^{m+n+k}(X/S). \end{array} \tag{1.6.13}$$

Proof. **For Item 1:** Note first that the following diagram is commutative :

$$\begin{array}{ccc} (X/S)^m \times (X/S)^n & \xrightarrow{\quad swap \quad} & (X/S)^n \times (X/S)^m \\ \downarrow & & \downarrow \\ \mathrm{Sym}^m(X/S) \times \mathrm{Sym}^n(X/S) & \xrightarrow{swap} & \mathrm{Sym}^n(X/S) \times \mathrm{Sym}^m(X/S). \end{array} \tag{1.6.14}$$

Further we observe that the map $(X/S)^m \times (X/S)^n \cong (X/S)^{m+n}$ followed by swapping the m first factors with the n last factors (that is followed by the permutation cycle $i \mapsto m+i \pmod{m+n}$) coincides with swapping the two factors of $(X/S)^m \times (X/S)^n$ followed by the isomorphism $(X/S)^n \times (X/S)^m \cong (X/S)^{m+n}$. Since the quotient $(X/S)^{m+n} \to \mathrm{Sym}^{m+n}(X/S)$ is Σ_{m+n} invariant, we conclude from these observations and the defining proeperty of $\sigma^{m,n}$ the desired commutativity.

For Item 2: A simple diagram chase shows that both compositions coincide with the morphism $\sigma^{(k,m,n)}$. ☐

From Lemma E.2.2 it follows that we can apply Construction E.2.3 to the $\sigma^{m,n}$ maps of Lemma 1.6.18 and make $\mathrm{Sym}^\bullet(X/S) := \coprod_{n\geq 0} \mathrm{Sym}^n(X/S)$ into a commutative monoid object in the category of S-schemes. This is how we will always view $\mathrm{Sym}^\bullet(X/S)$.

Proposition 1.6.19. *Let S be a Noetherian scheme and $X \to S$ a flat morphism of finite type and* AF*. Then the S-scheme* $\mathrm{Sym}^\bullet(X/S)$ *is locally of finite type, flat and* AF*.*

Proof. Follows easily from Lemma 1.6.14 and Lemma 1.5.29. ☐

Remark 1.6.20. There is a non-flat analogue of the scheme $\mathrm{Sym}^\bullet(X/S)$ called the *scheme of divided powers*. The scheme of divided powers and its relation to zero cycles is extensively studied in [Ryd08b].

1.7 Cycles on Noetherian schemes

In this section we again take our schemes to be separated in order to avoid potential pathologies.

Basic definitions

For a Noetherian scheme X we follow p.13 of [SV00] and use the following definitions and notations:

Definition 1.7.1.

1. $\mathrm{Cycl}(X)$ denotes the free abelian group generated by the points of (the Zariski topological space) of X.

2. $\mathrm{Cycl}^{\mathit{eff}}(X)$ is the free abelian monoid generated by the points of X.

3. An element of $\mathrm{Cycl}(X)$ (resp. of $\mathrm{Cycl}^{\mathit{eff}}(X)$) is called a *cycle* (resp. an *effective cycle*) of X.

4. We let $\mathrm{supp}(\mathcal{Z})$ denote the closure of the set of points on X which appear in the cycle $\mathcal{Z} \in \mathrm{Cycl}(X)$ with nonzero coefficients. We consider $\mathrm{supp}(\mathcal{Z})$ as a reduced closed subscheme of X.

5. If $\mathcal{Z} = \sum a_i z_i \in \mathrm{Cycl}(X)$ is a cycle, then we say that $z \in X$ is a *point* of $\mathcal{Z}$ if $z = z_i$ for some i occuring in the formal sum $\mathcal{Z}$ with nonzero a_i. The coefficient of a point of a given cycle is sometimes referred to as a *multiplicity* .

6. For a closed subscheme Z of X where the points $\xi_i, i = 1, \ldots, k$ are the generic points of Z (that is the generic points of the irreducible components of Z) we set

$$\mathrm{cycl}_X(Z) := \sum_{i=1}^{k} m_i \xi_i \in \mathrm{Cycl}^{\mathit{eff}}(X), \tag{1.7.1}$$

where $m_i := \mathrm{length}_{\mathcal{O}_{Z,\xi_i}}(\mathcal{O}_{Z,\xi_i})$. Each number m_i is a positive integer which is called the multiplicity of Z in the point ξ_i. This construction gives us a map from the set of closed subschemes of X to the abelian monoid $\mathrm{Cycl}^{eff}(X)$ which can be canonically extended to a homomorphism from the free abelian monoid generated by this set to $\mathrm{Cycl}^{eff}(X)$. We denote this homomorphism by cycl_X. .

7. A flat morphism $p : X \to S$ of Noetherian schemes induces a pullback map of cycles as follows: For a cycle $\mathcal{Z} = \sum n_i z_i$ on S denote by Z_i the closure of the point z_i which we consider as a closed integral subscheme in S and set

$$p^*(\mathcal{Z}) := \sum n_i \,\mathrm{cycl}_X(Z_i \underset{S}{\times} X). \tag{1.7.2}$$

In this way we get a homomorphism (flat-pullback) $p^* : \mathrm{Cycl}(S) \to \mathrm{Cycl}(X)$.

The following lemma is essentially [Ful98, Lemma 1.7.1].

Lemma 1.7.2 ([SV00, Lemma 2.3.1])**.** *Let $p : X \to S$ be a flat morphism of Noetherian schemes.*

1. *If Z is any closed subscheme of S then $p^*(\mathrm{cycl}_S(Z)) = \mathrm{cycl}_X(Z \times_S X)$.*

2. $\mathrm{supp}(p^*(\mathcal{Z})) = p^{-1}(\mathrm{supp}(\mathcal{Z}))$. *In particular the homomorphism p^* : $\mathrm{Cycl}(S) \to \mathrm{Cycl}(X)$ is injective provided that p is surjective.*

Cycles on algebraic schemes

Let $X \to \operatorname{Spec}(k)$ be a scheme of finite type over a field k. For any field extension k'/k base change induces a flat morphism of schemes $p : X_{k'} \to X$ which by Item 7 of Definition 1.7.1 gives rise to a homomorphism $p^* : \operatorname{Cycl}(X) \to \operatorname{Cycl}(X_{k'})$. The image of a cycle $\mathcal{Z} \in \operatorname{Cycl}(X)$ under this homomorphism will usually be denoted by $\mathcal{Z} \otimes_k k'$ or $\mathcal{Z}_{k'}$. For a finite field extension k'/k with Galois group $G = \operatorname{Gal}(k'/k)$, note that if $\sigma \in G$ then we get induced an automorphism $\rho(\sigma) : X_{k'} \to X_{k'}$ (in fact G gives a group action on $X_{k'}/\operatorname{Spec}(k)$) and this again induces a group action on $\operatorname{Cycl}(X_{k'})$, we let $\operatorname{Cycl}(X_{k'})^G$ denote the subgroup of cycles invariant under the action of G.

The following lemma is [SV00, Lemma 2.3.2]. Our proof contains several explanations which are omitted in loc.cit.

Lemma 1.7.3. *Let $X \to \operatorname{Spec}(k)$ be a scheme of finite type over a field k and let k'/k be a finite normal extension with Galois group G. If $\mathcal{Z}' \in \operatorname{Cycl}(X_{k'})^G$ then there is a unique cycle $\mathcal{Z} \in \operatorname{Cycl}(X)$ such that $[k' : k]_{\text{insep}} \mathcal{Z}' = \mathcal{Z}_{k'}$.*

Proof. The uniqueness of $\mathcal{Z}$ follows immediately from Lemma 1.7.2(2). To prove the existence note that the group $\operatorname{Cycl}(X_{k'})^G$ is generated by cycles of the form $\mathcal{Z}' = \sum_{\tau \in G/\operatorname{Stab}_G(z')} \tau(z')$, where z' is a point of $X_{k'}$. Let z be the image of z' in X and let Z be the closure of z in X. Consider now the commutative diagram where each square is a pullback.

$$\begin{tikzcd}
Z' \times_{\operatorname{Spec} k} \operatorname{Spec} k(z) \arrow[r] \arrow[d] & \operatorname{Spec}(k(z)) \arrow[d] \\
Z' := \operatorname{Spec}(k') \times_{\operatorname{Spec}(k)} Z \arrow[r] \arrow[d, hook] & Z \arrow[d, hook] \\
X_{k'} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(k') \arrow[r] & \operatorname{Spec}(k)
\end{tikzcd}$$

The points of $Z' \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k(z))$ correspond precisely to the irreducible components of Z', moreover by [Bou64, Ch 5, Sec. 2, n.2, Theorem 2] it follows that G acts transitively on the points of $Z' \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k(z))$, hence it follows that all the multiplicities appearing in $\operatorname{cycl}_{X_{k'}}(Z')$ are the same and equal to $\operatorname{length}_{\mathcal{O}_{Z',z'}}(\mathcal{O}_{Z',z'})$. Let l denote the number of irreducible components of Z' or equivalently the number of prime ideals of $k(z) \otimes_k k'$. From Lemma A.2.1 it follows that

$$\operatorname{length}_{\mathcal{O}_{Z',z'}}(\mathcal{O}_{Z',z'}) = \frac{\dim_{k(z)}(k(z) \otimes_k k')}{l \cdot [k(z') : k(z)]} = \frac{[k' : k]}{l \cdot [k(z') : k(z)]}$$

Since the map $\varphi : k'_{\mathrm{sep}} \otimes_k k(z) \to k' \otimes_k k(z)$ induces a universal homeomorphism it is clear that for any prime ideal $\mathfrak{p}$ of $k(z) \otimes_k k'$ with $q = \varphi^{-1}(\mathfrak{p})$ we have that the extension $k(q)/k(z)$ is separable since k'_{sep}/k is so ([GD67, Cor. (4.3.7)]), and since $k(z') \cong k(\mathfrak{p})/k(q)$ is purely inseparable it follows from Lemma 1.4.19 that we must have $k(q) \cong k(z')_{sep}$. Thus we get an isomorphism of $k(z)$-algebras

$$k(z) \otimes_k k'_{\mathrm{sep}} \cong k(z')^{\oplus l}_{\mathrm{sep}}$$

hence $l = \frac{[k':k]_{\mathrm{sep}}}{[k(z'):k(z)]_{\mathrm{sep}}}$. Thus

$$\mathrm{length}_{\mathcal{O}_{Z',z'}}(\mathcal{O}_{Z',z'}) = \frac{[k' : k]_{\mathrm{insep}}}{[k(z') : k(z)]_{\mathrm{insep}}}$$

Thus the cycle

$$\mathcal{Z} = [k' : k]_{\mathrm{insep}} z / \,\mathrm{length}(\mathcal{O}_{Z',z'}) = [k(z') : k(z)]_{\mathrm{insep}} z$$

has the required property. □

Corollary 1.7.4 ([SV00, Corollary 2.3.3])**.** *In the assumptions and notations of Lemma 1.7.3 denote by p the exponential characteristic of the field k. Then the homomorphism*

$$\mathrm{Cycl}(X)[1/p] \to (\mathrm{Cycl}(X_{k'})[1/p])^G$$

is an isomorphism.

Example 1.7.5. Consider the scheme $X = \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2+1))$ and the base change $X_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y]/(x^2+y^2+1))$. Consider the two points $p_1 = (x-i, y), p_2 = (x+i, y)$ on $X_{\mathbb{C}}$ and let $\mathcal{Z}'$ be the cycle

$$\mathcal{Z}' = p_1 + p_2$$

Note that this is invariant under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\mathrm{Cycl}(X_{\mathbb{C}})$. Let p be the image of p_1 in X, that is $p = (x^2+1, y)$ and set $\mathcal{Z} = p$. Then

$$\mathcal{Z} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{cycl}_{X_{\mathbb{C}}}(V(x^2+1, y)) = p_1 + p_2 = \mathcal{Z}'$$

Example 1.7.6. Consider the one point scheme $X = \mathrm{Spec}(\mathbb{F}_2(t)[x]/(x^2-t))$ and the base change $X' = X_{\mathbb{F}_2(\sqrt{t})}$. Note that the latter scheme is also a one point scheme. Let p respectively p' denote the point of X respectively of X'. Set $\mathcal{Z}' = p' \in \mathrm{Cycl}(X')$. A straightforward computation shows that $G = \mathrm{Gal}(\mathbb{F}_2(\sqrt{t})/\mathbb{F}_2(t)) = 0$ hence the cycle $\mathcal{Z}'$ is obviously invariant under the action of G. Set $\mathcal{Z} = p \in \mathrm{Cycl}(X)$ then one easily sees that $\mathcal{Z} \otimes_{\mathbb{F}_2(t)} \mathbb{F}_2(\sqrt{t}) = 2\mathcal{Z}'$ and thus $\mathcal{Z}$ is the unique cycle of Lemma 1.7.3.

Let $X \to \mathrm{Spec}(k)$ be a scheme of finite type over a field k. As stated on p.14 of [SV00] we have a direct sum decomposition $\mathrm{Cycl}(X) = \coprod \mathrm{Cycl}(X, r)$ where $\mathrm{Cycl}(X, r)$ is a subgroup of $\mathrm{Cycl}(X)$ generated by points of dimension r (meaning that their closure has dimension r). Furthermore since flatness of relative dimension 0 is stable under base change it follows that for a field extension k'/k the homomorphism

$$\mathrm{Cycl}(X) \to \mathrm{Cycl}(X_{k'})$$

preserves this decomposition.

The following Proposition tells us something about how multiplicities change when we pass to a larger field.

Proposition 1.7.7. *Let k be a field of exponential characteristic p and X an integral scheme of finite type over k. Let K denote the function field of X. For a field extension k'/k, denote by $\{X'_\alpha\}_\alpha$ the set of irreducible components of $X' := \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} X$ and x'_α the generic point of X_α.*

1. *If k'' is any field extension of k' we set $X'' := \mathrm{Spec}(k'') \times_{\mathrm{Spec}(k)} X$, $\{X''_\beta\}_\beta$ the set of irreducible components of X'' where x''_β denotes the generic point of X''_β. If x''_β lies over x'_α then we have*
$$\mathrm{length}(\mathcal{O}_{x''_\beta}) = \mathrm{length}(\mathcal{O}_{x'_\alpha}) \cdot \mathrm{length}(\mathcal{O}_{Z'',x''_\beta}), \qquad (1.7.3)$$
where $Z'' := \mathrm{Spec}(k'') \times_{\mathrm{Spec}(k')} X'_{\mathrm{red}}$. In particular if k'' is a separable extension of k' then we have $\mathrm{length}(\mathcal{O}_{x''_\beta}) = \mathrm{length}(\mathcal{O}_{x'_\alpha})$.
2. *The numbers $\mathrm{length}(\mathcal{O}_{x'_\alpha})$ are powers of p.*
3. *If the field k' is perfect then the numbers $\mathrm{length}(\mathcal{O}_{x'_\alpha})$ are all equal and only depend on the field extension K/k (and not on the considered field extension k'/k with k' perfect). In addition there exists a finite purely inseparable k_1/k such that if x_1 is the generic point of $X_1 = \mathrm{Spec}(k_1) \times_{\mathrm{Spec}(k)} X$ (this is an irreducible scheme) then $\mathrm{length}(\mathcal{O}_{x'_\alpha}) = \mathrm{length}(\mathcal{O}_{x_1})$ for every α.*

Proof. See [GD67, Prop. (4.7.3)]. □

Proper pushforward

The following definition is taken from the last paragraph of p.14 of [SV00].

Definition 1.7.8. Let S be a Noetherian scheme and $p : X \to S$ be a proper morphism of finite type. For any cycle $\mathcal{Z} = \sum n_i z_i \in \mathrm{Cycl}(X)$ set

$$p_*(\mathcal{Z}) = \sum n_i m_i p(z_i)$$

where m_i is the degree of the field extension $k_{z_i}/k_{p(z_i)}$ if this extension is finite and zero otherwise.

The proof of the following proposition is similar to Proposition 1.7 of [Ful98]

Proposition 1.7.9 ([SV00, Proposition 2.3.4]). *Consider a pull-back square of morphisms of finite type of Noetherian schemes*

$$\begin{tikzcd} \tilde{X} \arrow[r, "f_X"] \arrow[d, "\tilde{p}"] & X \arrow[d, "p"] \\ \tilde{S} \arrow[r, "f"] & S \end{tikzcd}$$

in which f is flat and p is proper. Then for any cycle $\mathcal{Z} \in \mathrm{Cycl}(X)$ we have

$$f^*(p_*(\mathcal{Z})) = \tilde{p}_*(f_X^*(\mathcal{Z})).$$

Cycles of codimension one

Proposition 1.7.10. *Let X be a Noetherian scheme and $W = \sum_i a_i z_i \in \mathrm{Cycl}^{eff}(X)$ such that the points z_i are of codimension 1 in X. Suppose that X is regular at the points z_i. Then there is exactly one closed subscheme Z of X satisfying the following two properties:*

1. $\mathrm{cycl}_X(Z) = W$ *and ;*
2. *Z has no embedded components.*

Proof. Let us first prove existence of Z. For each i let $\mathcal{I}_i$ denote the ideal sheaves of the integral closed subscheme $\overline{\{z_i\}}$ and let $\mathcal{I}$ be the ideal sheaf defined by $\mathcal{I} = \cap_i \mathcal{I}_i^{(a_i)}$ where $\mathcal{I}_i^{(a_i)}$ denotes the a_i'th symbolic power (See [Stacks, Tag 05G9] and [AM69, Chap.4, Exercise 13]). Let Z be the closed subscheme cut out by $\mathcal{I}$. It is clear that the associated points of Z are exactly the points z_i hence we have no embedded components, and moreover since a prime ideal p of a ring A has the property that $p^n A_p = p^{(n)} A_p$ and $\mathcal{O}_{X,z_i}$ is a discrete valuation ring it follows that we must have $\mathrm{cycl}_X(Z) = W$.

Let Z' be any other closed subscheme satisfying properties 1 and 2. In order to show that $Z' = Z$ we may assume that X is an affine scheme say $X = \mathrm{Spec}(A)$ and that Z' is cut out by an ideal I' of A. Since I' has no embedded points we have a primary decomposition of I' of the form $I' = \cap q_i$. By [AM69, Prop. 4.9] we have $I' \cdot \mathcal{O}_{X,z_i} = q_i \mathcal{O}_{X,Z_i}$ and the preimage of this ideal in A is exactly the ideal q_i. Now since $\mathcal{O}_{X,z_i}$ is a discrete valuation ring and Z' satisfies 1 we see that $q_i \mathcal{O}_{X,z_i}$ coincides with the a_i'th power of the maximal ideal of $\mathcal{O}_{X,z_i}$ and since this holds for all i we conclude that $Z = Z'$. □

Corollary 1.7.11. *Let X be a normal Noetherian scheme and $W = \sum_i a_i z_i \in \mathrm{Cycl}^{eff}(X)$ such that the points z_i are of codimension 1 in X. Suppose that every point $x \in X$ has a neighborhood U_x and a section $f \in \mathcal{O}_X(U_x)$ such that we have the equality*

$$\mathrm{cycl}_{U_x}(V(f)) = \sum_{z_i \in U_x} a_i z_i. \tag{1.7.4}$$

Then there is exactly one closed subscheme Z of X with no embedded components such that $\mathrm{cycl}_X(Z) = W$ and this closed subscheme Z is necessarily an effective Cartier divisor on X.

In particular if all the local rings $\mathcal{O}_{X,x}$ of X are unique factorization domains[9] *then any effective Weil divisor on X is of the form $\mathrm{cycl}_X(Z)$ for a unique effective Cartier divisor Z on X.*

Proof. Since normal schemes are regular in codimension one we have by Proposition 1.7.10 a unique closed subscheme Z of X with no embedded components such that $\mathrm{cycl}_X(Z) = W$. To see that this is an effective Cartier divisor note first that

$$cycl_{U_x}(V(f)) = \mathrm{cycl}_{U_x}(Z \cap U_x), \tag{1.7.5}$$

and we may shrink the U_x so that they may be taken to be affine integral schemes. By [Stacks, Tag 031T] it follows then that $V(f)$ does not have any embedded components hence we conclude that $V(f) = Z \cap U_x$, which completes the proof.

The last statement follows easily from the fact that every prime ideal of height 1 in a Noetherian unique factorization domain is necessarily principal ([Stacks, Tag 0AFT]). $\square$

Lemma 1.7.12. *Let X be a smooth equidimensional algebraic scheme of dimension r over a field k and Z a closed subscheme of pure dimension $r-1$. If there is a field extension K/k such that the base change Z_K is an effective Cartier divisor on X_K then Z is an effective Cartier divisor on X.*

Proof. If Z_K is an effective Cartier divisor of X_K then since X_K is smooth the scheme Z_K is necessarily Cohen-Macaulay. Thus by [Stacks, Tag 045U] it follows that Z is also a Cohen-Macaulay scheme. Hence Z has no embedded components and all generic points of Z are of codimension 1 in X. By Corollary 1.7.11 we conclude that Z is an effective Cartier divisor on X. $\square$

1.8 Limits of schemes

When attacking a problem concerning a Noetherian scheme we can sometimes instead work with its normalization. The problem is however that in general the normalization of a Noetherian scheme need not be Noetherian, and even if it is the normalization morphism might not be finite, see for instance [Nag62, Example 5, Appendix A.1, page 207] and [Nag62, Example 3, Appendix A.1, page 203] respectively. The theory of directed limits of schemes provides tools to overcome such problems by means of "approximating" our schemes or morphisms by ones which are better behaved. Our main reference is [Stacks, Tag 01YT] which again is largely based on [GD67].

[9]This is the case if X is for instance a regular scheme ([Stacks, Tag 0AG0]).

Setup

Definition 1.8.1. Let I be a set and let $\leq$ be a binary relation on I.

1. We say $\leq$ is a *preorder* if it is transitive (if $i \leq j$ and $j \leq k$ then $i \leq k$) and reflexive ($i \leq i$ for all $i \in I$).

2. A *preordered set* is a set endowed with a preorder.

3. A *directed set* is a preordered set $(I, \leq)$ such that I is not empty and such that $\forall i, j \in I$, there exists $k \in I$ with $i \leq k, j \leq k$.

It is customary to drop the $\leq$ from the notation when talking about preordered sets, that is, one speaks of the preordered set I rather than of the preordered set $(I, \leq)$. Given a preordered set I the symbol $\geq$ is defined by the rule $i \geq j \Leftrightarrow j \leq i$ for all $i, j \in I$.

Given a preordered set I we can construct a category: the objects are the elements of I, there is exactly one morphism $i \to i'$ if $i \leq i'$, and otherwise none.

Definition 1.8.2. Let I be a preordered set considered as a category $\mathcal{I}$ and $\mathcal{C}$ any category.

1. A *system* (of $\mathcal{C}$) over I is a diagram $M : \mathcal{I} \to \mathcal{C}$.

2. An *inverse system* (of $\mathcal{C}$) over I is a diagram $M : \mathcal{I}^{op} \to \mathcal{C}$.

For short we will often write $(M_i, f_{ii'})$ to denote an (inverse) system, where $M_i := M(i)$ and $f_{ii'} = M(i \to i')$. We will call the $f_{ii'}$ *transition maps.*

Lemma 1.8.3 ([Stacks, Tag 01YW])**.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \operatorname{Spec}(\operatorname{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O}_{S_i})$.*

Lemma 1.8.4 ([Stacks, Tag 01YX])**.** *Let I be a directed set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*

2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*

$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0).$$

in the category of schemes.

Descending properties

The following Lemma is [Stacks, Tag 01YZ].

Lemma 1.8.5. *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms. For some $0 \in I$ suppose that T is a scheme over S_0. Then*

$$T \underset{S_0}{\times} S = \lim_{i \geq 0} T \underset{S_0}{\times} S_i. \tag{1.8.1}$$

Proof. Limits commute with fiber products. □

Lemma 1.8.6. *If all the schemes S_i are nonempty and quasi-compact, then the limit $S = \lim_i S_i$ is nonempty.*

Proof. This is [Stacks, Tag 01Z2]. □

Lemma 1.8.7. *Let $S = \lim S_i$ be the limit of a directed inverse system of schemes with affine transition morphisms. Suppose that all the schemes S_i are quasi-compact and quasi-separated. Then given any quasi-compact open $V \subset S$ there is some $i \in I$ and a quasi-compact open $V_i \subset S_i$ such that $V = f_i^{-1}(V_i)$ where $f_i : S \to S_i$ is the projection from the limit.*

Proof. See [Stacks, Tag 01Z4]. □

Nagata schemes

We briefly recall that a domain R is according to [GD67, Ch. 0, (23.1.1)] called *Japanese* if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called *universally Japanese* if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called *Nagata* if it is Noetherian and R/p is Japanese for every prime p of R. One has that a ring R is *Nagata* if and only if every finite type R-algebra is Nagata if and only if R is Noetherian and universally Japanese ([Stacks, Tag 0334]). A scheme X is *Nagata* if for any affine open U of X the ring $\mathcal{O}_X(U)$ is Nagata. The normalization of a Nagata scheme X is necessarily finite over X ([Stacks, Tag 035S]) and [Stacks, Tag 0AVK] tells us that if X is an integral Nagata scheme then the normalization of X in a finite field extension of its function field is finite over X. Finally if R is either $\mathbb{Z}$ or a field and $X \to \mathrm{Spec}(R)$ is a morphism locally of finite type, then [Stacks, Tag 035B] yields that X must necessarily be Nagata. [Stacks, Tag 01ZA] tells us now that any quasi-compact and quasi-separated scheme is a directed limit of Nagata schemes with affine transition maps, however this fact will not be used in this thesis.

Limits and morphisms of finite presentation

Proposition 1.8.8. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *The morphism f is locally of finite presentation.*
2. *For any directed set Λ, and any inverse system $(T_\lambda, f_{\lambda\lambda'})$ of Y-schemes over Λ with each $f_{\lambda\lambda'}$ affine and every T_λ quasi-compact and quasi-separated as a scheme, we have*

$$\mathrm{Mor}_Y(\lim_\lambda T_\lambda, X) = \mathrm{colim}_\lambda \mathrm{Mor}_Y(T_\lambda, X)$$

Proof. See [GD67, Proposition 8.14.2] or [Stacks, Tag 01ZC] □

Approximating integral morphisms by finite morphisms

Lemma 1.8.9. *Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.*

Proof. [Stacks, Tag 09YZ] □

Lemma 1.8.10. *Let S be an integral Noetherian scheme with function field K and let $\nu : S^n \to S$ denote its normalization. The following statements hold true:*

1. *The normalization $\nu : S^n \to S$ can be expressed as a directed limit*
$$S^n = \lim_\lambda S_\lambda \tag{1.8.2}$$
with $p_\lambda : S^n \to S_\lambda$ integral and surjective and $S_\lambda \to S$ finite. In particular all the S_λ are integral Noetherian schemes with function field K.
2. *Given $s' \in S^n$ then in the notation of (1) there exists some λ and some ${s'}_\lambda \in S_\lambda$ such that s' is the only point lying over ${s'}_\lambda$ or in other words*
$$p_\lambda^{-1}(\{{s'}_\lambda\}) = \{s'\} \tag{1.8.3}$$

Proof. The first claim follows almost immediately from [Stacks, Tag 0817] (see also Lemma 1.8.9).

For the second claim set $s = \nu(s')$ and let $\mathrm{Spec}(A)$ be an open affine of S containing the point s, then clearly $s' \in S^n \times_S \mathrm{Spec}(A) = \mathrm{Spec}(\overline{A})$ where $\overline{A}$ denotes the normalization of the ring A. By [HS06, Lemma 4.8.4] there exists a finitely generated A algebra C with generators say $r_1, \dots, r_n \in C$ such that $C \subset \overline{A}$ and there exists an ${s'}_\lambda \in \mathrm{Spec}(C)$ such that s' is the only point of $\mathrm{Spec}(\overline{A})$ lying over ${s'}_\lambda$. By construction of the schemes occuring in the limit of (1) it is thus enough to construct a finite quasi-coherent $\mathcal{O}_S$-sub-algebra $\mathcal{C}$ of $\nu_*\mathcal{O}_{S^n}$ such that $C = \Gamma(\mathrm{Spec}(A), \mathcal{C})$. This is done mutatis mutandis as in the proof of [Stacks, Tag 01PF]. □

Chapter 2

Relative Cycles

In this chapter we will familiarize ourselves with relative cycles as developed in [SV00]. In doing so we will follow op.cit. closely, sometimes providing more explanations and on other occasions omit results and/or proofs that can be found there.

Towards the end of the chapter we briefly recall Kollár's theory of families of well defined proper algebraic cycles and explain how they relate to Suslin-Voevodsky's relative cycles. Furthermore we will also study the locus where a relative cycle becomes effective and vanishes, something which as far as we know has not been considered in the literature.

In this chapter every scheme is assumed to be separated.

2.1 Relative cycles

Fat points

Definition 2.1.1 ([SV00, Def.3.1.1])**.** Let S be a Noetherian scheme, k a field and $x : \operatorname{Spec}(k) \to S$ be a k-point of S. A *fat point* over x is a triple (x_0, x_1, R), where R is a discrete valuation ring and $x_0 : \operatorname{Spec}(k) \to \operatorname{Spec}(R)$, $x_1 : \operatorname{Spec}(R) \to S$ are morphisms such that

1. $x = x_1 \circ x_0$

2. The image of x_0 is the closed point of $\operatorname{Spec}(R)$.

3. x_1 takes the generic point of $\operatorname{Spec}(R)$ to a generic point of S.

Usually we will abbreviate the notation (x_0, x_1, R) to (x_0, x_1).

The following algebraic lemma shows us how to cook up fat points over suitably large fields:

Lemma 2.1.2 ([GD61, (7.1.7)])**.** *Suppose $(A, \mathfrak{m})$ is a local Noetherian integral domain which is not a field, and denote by K the field of fractions of A, let L*

be any finite field extension of K. Then there exists a discrete valuation ring R with field of fractions L such that R dominates A.

Proof. The proof we give here is a rather faithful translation of the French proof.

We only prove the case $L = K$. Let $x_1, \dots, x_n$ be non-zero generators of the maximal ideal $\mathfrak{m}$ of A. Consider the subring

$$B := A[x_2/x_1, \dots, x_n/x_1] \subset K.$$

It is clear that B is Noetherian. Note also that the extended ideal $\mathfrak{m}B = (x_1)B$ is principal. Let $\mathfrak{p}$ be a minimal prime containing x_1B, then the local ring

$$B_{\mathfrak{p}}$$

is a one-dimensional Noetherian ring. It is clear that $B_{\mathfrak{p}}$ dominates A. Let C be the integral closure of $B_{\mathfrak{p}}$ in K, then by the Krull-Akizuki theorem we have that C is Noetherian([Stacks, Tag 00PG]) and since C is integral over $B_{\mathfrak{p}}$ it follows that C is one-dimensional. Now let η be any maximal ideal of C, then $R = C_\eta$ is a DVR dominating $B_{\mathfrak{p}}$ hence also A. $\square$

Corollary 2.1.3 ([SV00, Lemma.3.1.4])**.** *Let S be a Noetherian scheme, η a generic point of S and s be a point in the closure of η. Let further L be an extension of finite type of the field of functions on S in η. Then there is a discrete valuation ring R and a morphism $f : \mathrm{Spec}(R) \to S$ such that the following conditions hold:*

1. *f maps the generic point of $\mathrm{Spec}(R)$ to η and the field of functions of R is isomorphic to L over $k(\eta)$.*

2. *f maps the closed point of $\mathrm{Spec}(R)$ to s.*

Example 2.1.4. Let k be a field and consider the scheme $\mathbb{A}^n_k$ and let $\mathfrak{m}$ be the origin $\mathfrak{m} = (x_1, \dots, x_n) \in \mathrm{Spec}(k[x_1, \dots, x_n]) = \mathbb{A}^n_k$. In this case we can explicitly produce a map $\varphi : k[x_1, \dots, x_n] \to R$ where R is a discrete valuation ring such that $\mathrm{Spec}(\varphi)$ takes the closed point of $\mathrm{Spec}(R)$ to $\mathfrak{m}$ and the generic point to the generic point of $\mathbb{A}^n_k$. The construction of R is as follows: Consider the ring

$$B := k[x_1, \dots, x_n, u_2, \dots, u_n]/(x_1u_2 - x_2, x_1u_3 - x_3, \dots, x_1u_n - x_n).$$

Since $B/x_1B = k[u_2, \dots, u_n]$ it follows that x_1B is a prime ideal of B. Furthermore it is easily checked that B is an integral domain. Set $R = B_{(x_1B)}$. By Krull's Hauptidealsatz it follows that $\dim R = 1$, and since this is a local Noetherian domain of dimension one where the maximal ideal is principal, it follows that R is a discrete valuation ring. We claim that the canonical map $k[x_1, \dots, x_n] \to B$ is an injection. If we have some nonzero polynomial

$f \in k[x_1, \dots, x_n]$ such that the image of f in B is zero, then we have polynomials $g_2, \dots, g_n \in k[x_1, \dots, x_n, u_2, \dots, u_n]$ such that

$$g := \sum_{i=2}^{n} g_i(x_1 u_i - x_i) = f.$$

By possibly passing to the algebraic closure of k, we can find a tuple $(a_1, \dots, a_n) \in \bar{k}^n$ with $a_1 \neq 0$ such that $f(a_1, \dots, a_n) \neq 0$. But then $g(a_1, \dots, a_n, \frac{a_2}{a_1}, \frac{a_3}{a_1}, \dots, \frac{a_n}{a_1}) = 0$, which gives a contradiction. Hence the morphism $\varphi : k[x_1, \dots, x_n] \to R$ is an injection, thus the generic point of $\mathrm{Spec}(R)$ lies over the generic point of $\mathbb{A}^n_k$. Furthermore as $\varphi^{-1}(xR) \supset \mathfrak{m}$ and $1 \notin \varphi^{-1}(xR)$ it follows that $\varphi^{-1}(xR) = \mathfrak{m}$ thus the closed point of $\mathrm{Spec}(R)$ is mapped to the origin of $\mathbb{A}^n_k$.

The following example illustrates that there does not always exist a fat point over a given k-point.

Example 2.1.5. If S is any Noetherian scheme with normalization $S^n \to S$ such that there is a point $s \in S$ with the property that all points $s' \in S^n$ lying over s induce non-trivial extensions of residue fields $k(s) \to k(s')$. Then it follows easily from Proposition 1.3.11 that there cannot be a fat point over the point $\mathrm{Spec}(k(s)) \to S$.

For a concrete example of such a scheme S in characteristic 0 consider the plane curve $S = V(x^2 + y^2 - y^3) \subset \mathbb{A}^2_{\mathbb{R}}$.[1] The ring homomorphism

$$\mathbb{R}[x, y]/(x^2 + y^2 - y^3) \to \mathbb{R}[T(T^2 + 1), T^2 + 1]$$

given by $x \mapsto T(T^2 + 1)$, $y \mapsto (T^2 + 1)$ is in fact an isomorphism and the normalization of the latter ring is obviously $\mathbb{R}[T]$. Hence the normalization is the induced map $S' = \mathbb{A}^1_{\mathbb{R}} \to S$. Moreover the only point of S' lying over the origin $s = (x, y) \in C$ is $s' = (T^2 + 1) \in S'$ whose residue field is $\mathbb{C}$ while s has residue field $\mathbb{R}$.

Lemma 2.1.6 ([GD67, (2.8.5)])**.** *Suppose that Y is a regular irreducible locally Noetherian scheme of dimension 1 with generic point η, and suppose $f : X \to Y$ is a morphism. Let X_η denote the generic fiber of the morphism f and $i : X_\eta \to X$ be the projection. Then for any closed subscheme Z' of X_η there exists a unique closed subscheme $\overline{Z'}$ of X which is flat over Y and satisfies the equality $i^{-1}(\overline{Z'}) = Z'$.*

Proof. We only explain how the closed subscheme $\overline{Z'}$ is constructed following loc.cit. If $\mathcal{I}'$ is the ideal sheaf of Z' then

$$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{i^\#} i_* \mathcal{O}_{X_\eta} \to i_*(\mathcal{O}_{X_\eta}/\mathcal{I}'))$$

[1] This example has been given on mathoverflow by J. Starr.

is the ideal sheaf of $\overline{Z'}$, and we have $\mathcal{I}' = i^*(\mathcal{I})\mathcal{O}_{X_\eta}$. The closed subscheme $\overline{Z'}$ can also be described as the scheme theoretic image of the composition

$$Z' \hookrightarrow X_\eta \xrightarrow{i} X$$

which has as underlying set the closure of the set theoretic image of the aforementioned composition, justifying the notation $\overline{Z'}$. □

Corollary 2.1.7 ([SV00, Lemma 3.1.2])**.** *Let S be a Noetherian scheme, $X \to S$ be a scheme over S and Z be a closed subscheme in X. Let further R be a discrete valuation ring and $f : \operatorname{Spec}(R) \to S$ be a morphism. Then there exists a unique closed subscheme $\phi_f(Z)$ in $Z \times_S \operatorname{Spec}(R)$ such that:*

1. *The closed embedding $\phi_f(Z) \to Z \times_S \operatorname{Spec}(R)$ is an isomorphism over the generic point of* $\operatorname{Spec}(R)$.

2. *$\phi_f(Z)$ is flat over* $\operatorname{Spec}(R)$.

Proof. In Lemma 2.1.6 replace X with $Z \times_S \operatorname{Spec}(R)$, Y with $\operatorname{Spec}(R)$ and Z' with the generic fiber of $Z \times_S \operatorname{Spec}(R) \to \operatorname{Spec}(R)$, then $\phi_f(Z)$ is the scheme $\overline{Z'}$ which is the scheme theoretic image of the generic fiber of the aforementioned map. □

Lemma 2.1.8. *Let S be a Noetherian scheme, $X \to S$ be a scheme over S and Z be a closed subscheme in X. Assume that there exists a blow-up $S' \to S$ of S with center S_0 satisfying the two following properties:*

1. *$S' \to S$ induces a bijection of the generic points of S' and S.*

2. *The strict transform of Z denoted $\tilde{Z}$ is flat over S'.*

Then for any discrete valuation ring R and morphism $f : \operatorname{Spec}(R) \to S$ mapping the generic point of $\operatorname{Spec}(R)$ to a point in $S \setminus S_0$, the map $f : \operatorname{Spec}(R) \to S$ factors uniquely through S' and we have the equality

$$\phi_f(Z) = \operatorname{Spec}(R) \underset{S'}{\times} \tilde{Z}$$

Proof. It follows from the valuative criterion of properness ([GD61, (7.3.8)]) that $f : \operatorname{Spec}(R) \to S$ factors through S'.

Consider now the commutative diagram where each square is a pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(R) \times_{S'} \tilde{Z} \arrow[r] \arrow[d, hook] & \tilde{Z} \arrow[d, hook] & \\
\mathrm{Spec}(R) \times_S Z \arrow[r] \arrow[d, hook] & S' \times_S Z \arrow[r] \arrow[d, hook] & Z \arrow[d, hook] \\
\mathrm{Spec}(R) \times_S X \arrow[r] \arrow[d] & S' \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(R) \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

To prove the desired equality it is enough to check that $\mathrm{Spec}(R) \times_{S'} \tilde{Z}$ satisfies (1) and (2) of Corollary 2.1.7. Since $\tilde{Z} \to S'$ is flat by assumption it follows that $\mathrm{Spec}(R) \times_{S'} \tilde{Z} \to \mathrm{Spec}(R)$ is flat as well. Further we have by Lemma 1.2.2 that the strict transform $\tilde{Z} \to S' \times_S Z$ is an isomorphism over $S' \times_S (S \setminus S_0)$ and since the generic point of $\mathrm{Spec}(R)$ is mapped to the aforementioned open set it follows that $\mathrm{Spec}(R) \times_{S'} \tilde{Z} \to \mathrm{Spec}(R) \times_S Z$ is an isomorphism over the generic point of $\mathrm{Spec}(R)$. $\square$

Following [SV00, p.16] we use the following notation:

Notation 2.1.9. Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and Z be a closed subscheme of X. If (x_0, x_1) is a fat point over a k-point x of S we denote by $(x_0, x_1)^*(Z/S)$ the cycle on $X \times_S \mathrm{Spec}(k)$ associated with the closed subscheme $\phi_{x_1}(Z) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ or in other words

$$(x_0, x_1)^*(Z/S) := \mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(\phi_{x_1}(Z) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k)).$$

If $\mathcal{Z} = \sum m_i z_i$ is any cycle on X we denote by $(x_0, x_1)^*(\mathcal{Z})$ the cycle $\sum m_i (x_0, x_1)^*(Z_i)$ where Z_i is the closure of the point z_i (considered as a reduced closed subscheme of X) .

Example 2.1.10. Let S be a regular curve, $X \to S$ any morphism and let Z be a closed integral subscheme of X such that the generic point of Z lies over a generic point of S. Then it follows that Z is flat over S, thus if $f : \mathrm{Spec}(R) \to S$ is a morphism from the spectrum of a discrete valuation ring, then $\phi_f(Z) = Z \times_S \mathrm{Spec}(R)$. In particular if $x : \mathrm{Spec}(k) \to S$ is any k-point of S and (x_0, x_1) is any fat point over x, then we have $(x_0, x_1)^*(Z) = \mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z \times_S \mathrm{Spec}(k))$.

Relative cycles

Definition 2.1.11 (Relative cycles, [SV00, Def.3.1.3])**.** Let S be a Noetherian scheme and $X \to S$ be a scheme of finite type over S. A *relative cycle* on X over S is a cycle $\mathcal{Z} = \sum m_i z_i$ on X satisfying the following requirements:

1. The points z_i lie over generic points of S.

2. For any field k, k-point x of S and a pair of fat points $(x_0, x_1), (y_0, y_1)$ of S over x one has:
$$(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z})$$

We say that $\mathcal{Z} = \sum n_i z_i$ is a *relative cycle of dimension* r if each point z_i has dimension r in its fiber over S. We denote the corresponding abelian groups by $\mathrm{Cycl}(X/S, r)$.

We say that $\mathcal{Z}$ is an *equidimensional relative cycle of dimension* r if $\mathrm{Supp}(\mathcal{Z})$ is equidimensional of dimension r over S which by Corollary 1.1.19 is equivalent to requiring that $\mathrm{Supp}(z_i)$ is equidimensional of dimension r over S for each of the points z_i occuring in the sum. We denote the corresponding abelian groups by $\mathrm{Cycl}_{equi}(X/S, r)$.

We say that $\mathcal{Z}$ is a *proper relative cycle* if $\mathrm{Supp}(\mathcal{Z})$ is proper over S. We denote the corresponding abelian groups by $\mathrm{PropCycl}(X/S, r)$ and $\mathrm{PropCycl}_{equi}(X/S, r)$.

We will also use the notations $\mathrm{Cycl}^{eff}(X/S, r)$, $\mathrm{PropCycl}^{eff}(X/S, r)$ etc. for the corresponding abelian monoids of effective relative cycles.

Remark 2.1.12 ([SV00, p.17])**.** It is clear from the definition that $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)$ if and only if $\mathcal{Z} \in \mathrm{Cycl}(X_{\mathrm{red}}/S_{\mathrm{red}}, r)$.

For future reference we now make the following useful observation concerning 2.1.11.

Lemma 2.1.13. *Let S be a Noetherian scheme and $f : X \to S$ be a scheme of finite type over S. Let z be a point of X lying over a generic point of S and let Z be the closure of z in X (with induced reduced subscheme structure). Then the closure of z in the fiber of $f(z)$ is isomorphic to $Z \times_S \mathrm{Spec}(k(f(z)))$. In particular a point z of X has dimension r in its fibre if and only if $Z \times_S \mathrm{Spec}(k(f(z)))$ is an r-dimensional scheme.*

Proof. Let S_z denote the irreducible component of S with generic point $f(z)$. It is straightforward to see that
$$Z \underset{S}{\times} \mathrm{Spec}(k(f(z))) \cong Z \underset{S_z}{\times} \mathrm{Spec}(k(f(z)))$$
The latter can easily be seen to be an integral scheme by reducing to the affine case. It is furthermore straightforward to see that the underlying set of $Z \times_S \mathrm{Spec}(k(f(z)))$ and that of the closure of z in $X \times_S \mathrm{Spec}(k(f(z)))$ coincide, thus the closure of z in $X \times_S \mathrm{Spec}\, k(f(z))$ and $Z \times_S \mathrm{Spec}\, k(f(z))$ are reduced subschemes with the same underlying sets, thus they are equal. $\square$

Corollary 2.1.14. *Let S be a Noetherian scheme and $f : X \to S$ be a scheme of finite type over S. Let z be a point of X lying over a generic point of*

S with dimension r in its fiber. Let Z denote the closure of z in X. Let $\operatorname{Spec} L \to \operatorname{Spec} k(f(z))$ *be a morphism from any field L. Then the scheme* $\operatorname{Spec}(L) \times_S Z$ *is equidimensional of dimension r.*

Example 2.1.15. Suppose that S is a Noetherian scheme and $f : X \to S$ is a morphism of finite type. Of course if a point z of X has dimension r in its fiber, this does not mean that the closure of the point in X is of dimension r. Consider for instance $S = \operatorname{Spec}(\mathbb{Z})$ and let $X = \operatorname{Spec}(\mathbb{Z}[x])$ then the canonical inclusion of rings gives us the map $X \to S$. Consider the point $(x) \in X$. The closure of (x) with induced reduced subscheme structure is isomorphic to $\operatorname{Spec}(\mathbb{Z})$ hence the closure of (x) is a one dimensional scheme, however the closure of (x) in its fiber is isomorphic to $\operatorname{Spec}(\mathbb{Q})$ which is zero dimensional.

Before moving on we also show how we can find fat points to tell different cycles apart.

Lemma 2.1.16. *Let S be a Noetherian scheme and $X \to S$ be a scheme of finite type over S. Suppose that $\mathcal{Z} = \sum a_i z_i \in \operatorname{Cycl}(X/S, r)$ is such that all the z_i lie over the same generic point η of S. Then there exists a fat point (x_0, x_1) over the canonical point* $\operatorname{Spec} k(\eta) \to S$ *such that*

$$(x_0, x_1)^*(\mathcal{Z}) = \sum a_i z_i \in \operatorname{Cycl}(X \times_S \operatorname{Spec} k(\eta))$$

Proof. Let $R = k(\eta)[t]_{(t)}$ where t is an independent variable, then R is a discrete valuation ring containing the field $k(\eta)$ with residue field $k(\eta)$ thus we have canonical maps $\operatorname{Spec} k(\eta) \to \operatorname{Spec} R \to \operatorname{Spec} k(\eta)$ and so in turn canonical maps

$$\operatorname{Spec} k(\eta) \overset{x_0}{\to} \operatorname{Spec} R \overset{x_1}{\to} S$$

it is clear that (x_0, x_1) has the desired property. □

Corollary 2.1.17. *Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S. If $\mathcal{Z}_1 = \sum a_i z_i$, $\mathcal{Z}_2 = \sum b_j z_j$ are two different cycles in* $\operatorname{Cycl}(X/S, r)$. *Then there exists a field k and a k-point $x : \operatorname{Spec}(k) \to S$ and a fat point (x_0, x_1) over x such that*

$$(x_0, x_1)^*(\mathcal{Z}_1) \neq (x_0, x_1)^* \mathcal{Z}_2$$

Proof. Suppose that $f : \operatorname{Spec}(R) \to S$ is a morphism from the spectrum of a discrete valuation ring mapping the generic point of $\operatorname{Spec}(R)$ to a generic point η_f of S. If Z is a closed subscheme of X disjoint from the fiber of η_f then since flat morphisms of finite type are open, it follows that $\phi_f(Z) = \emptyset$. Hence by Corollary 2.1.3 we may assume that any generic point of S which is in the image of $\mathcal{Z}_1$ is also in the image of $\mathcal{Z}_2$, in fact we can assume that all the points of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ lie over the same generic point η of S. By Lemma 2.1.16 we are now done. □

The following result is [SV00, Cor.3.1.6]. We shall give a different proof than in loc.cit.

Proposition 2.1.18. *Let k be a field and $X \to \operatorname{Spec}(k)$ be a scheme of finite type over k. Then the group $\operatorname{Cycl}(X/\operatorname{Spec}(k), r)$ is the free abelian group generated by points of dimension r on X, i.e one has*

$$\operatorname{Cycl}(X/\operatorname{Spec}(k), r) = \operatorname{Cycl}_{equi}(X/\operatorname{Spec}(k), r) = \operatorname{Cycl}(X, r).$$

Proof. Let R be a discrete valuation ring and $f : \operatorname{Spec}(R) \to \operatorname{Spec}(k)$ be a morphism. Let Z be a closed subscheme of X. Since Z is flat over $\operatorname{Spec}(k)$ we have that $\operatorname{Spec}(R) \times_{\operatorname{Spec}(k)} Z$ is flat over $\operatorname{Spec}(R)$ thus automatically satisfying the properties of $\phi_f(Z)$ hence they are equal. Hence it follows that if L is any field and $x : L \to \operatorname{Spec}(k)$ is any L-point of $\operatorname{Spec}(k)$ then for any fat point (x_0, x_1) over x we have

$$(x_0, x_1)^*(Z) = \operatorname{cycl}_{\operatorname{Spec}(L) \times_{\operatorname{Spec}(k)} X}(\operatorname{Spec}(L) \underset{\operatorname{Spec}(k)}{\times} Z)$$

whence Item 2 of Definition 2.1.11 is automatically satisfied. □

Lemma 2.1.19. *Let S be a Noetherian scheme and $X \to S$ a scheme of finite type over S. Then the following assertions hold true:*

1. *If Z is a subscheme of X such that the map $Z \to X \to S$ factors through $S_{\mathrm{red}} \to S$ and $T \to S_{\mathrm{red}} \to S$ is a morphism of schemes. Then we have*
$$Z \underset{S_{\mathrm{red}}}{\times} T = Z \underset{S}{\times} T. \tag{2.1.1}$$

2. *If $\mathcal{Z} = n_i z_i$ is a cycle on X such that the points z_i lie over generic points of S and are of dimension r in the corresponding fibers. Then we have*
$$\mathcal{Z} \in \operatorname{Cycl}(X/S, r) \text{ if and only if } \mathcal{Z} \in \operatorname{Cycl}(X_{\mathrm{red}}/S_{\mathrm{red}}, r). \tag{2.1.2}$$

Proof. The first statement follows easily from the fact that closed embeddings are monomorphisms in the category of schemes ([Stacks, Tag 01L7]). The second statement follows easily from the first assertion and the definitions. □

Observation 2.1.20 ([SV00, p.17])**.** Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z} = \sum n_i z_i$ be a cycle on X such that the points z_i lie over generic points of S and are of dimension r in the corresponding fibers. Let Z_i denote the closure of z_i considered as a closed integral subscheme of X. By Lemma 2.1.19 we see that in order to check if $\mathcal{Z} \in \operatorname{Cycl}(X/S, r)$ we may only consider the schemes X_{red} and S_{red}. Then by generic flatness we have that the schemes $Z_i \to S_{\mathrm{red}}$ are flat over generic points of S_{red} and we can find a dense subset U of S_{red} which all the Z_i are flat over. Thus by Theorem 1.2.3 one can find a blow-up $S' \to S_{\mathrm{red}}$ such that the proper transforms $\tilde{Z}_i$ are flat over S'.

We can now formulate the following useful criterion. Due to this results major importance we have included the proof found in [SV00].

Proposition 2.1.21 ([SV00, Proposition 3.1.5])**.** *Under the assumptions of Observation 2.1.20 the following conditions are equivalent:*

1. $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)$.

2. *If* $x : \mathrm{Spec}(k) \to S$ *is any geometric point of* S *and*

$$x'_1, x'_2 : \mathrm{Spec}(k) \to S'$$

is a pair of its liftings to S' *then the cycles* $\mathcal{W}_1, \mathcal{W}_2$ *on* $X_x = X \times_S \mathrm{Spec}(k)$ *given by the formulae*

$$\mathcal{W}_1 = \sum_{i=1}^{k} n_i \,\mathrm{cycl}_{X_x}(\tilde{Z}_i \underset{x'_1}{\times} \mathrm{Spec}(k))$$

$$\mathcal{W}_2 = \sum_{i=1}^{k} n_i \,\mathrm{cycl}_{X_x}(\tilde{Z}_i \underset{x'_2}{\times} \mathrm{Spec}(k))$$

coincide.

Proof. $(1 \Rightarrow 2)$: The geometric points x, x'_1, x'_2 give us set theoretical points $s \in S$, $s'_1, s'_2 \in S'$ such that s'_1, s'_2 lie over s. We may assume that s (and hence also s'_1, s'_2) is not generic. Using Corollary2.1.3 we construct discrete valuation rings R'_i and morphisms $\mathrm{Spec}(R'_i) \to S'$ which map the closed point of $\mathrm{Spec}(R'_i)$ to s'_i and the generic point of $\mathrm{Spec}(R'_i)$ to a generic point of S'. Denote the residue fields of R'_i by k'_i. One checks easily that the scheme $(\mathrm{Spec}(k'_1) \times_S \mathrm{Spec}(k'_2)) \times_{S' \times_S S'} \mathrm{Spec}(k)$ is not empty. Choosing any geometric L-point of this scheme for a field L we get a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(k'_1) \arrow[r] & \mathrm{Spec}(R'_1) \arrow[r] & S' \arrow[dr] & \\
\mathrm{Spec}(L) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{Spec}(k) \arrow[ur, "x'_1"] \arrow[rr, "x"] \arrow[dr, "x'_2"'] & & S \\
 & \mathrm{Spec}(k'_2) \arrow[r] & \mathrm{Spec}(R'_2) \arrow[r] & S' \arrow[ur] &
\end{tikzcd}$$

Thus we get a geometric point $\mathrm{Spec}(L) \to S$ and two fat points $\mathrm{Spec}(L) \to \mathrm{Spec}(R'_i) \to S$ over it. From commutativity of the aforementioned diagram and Lemma 2.1.8 it follows that the pullbacks of the cycle $\mathcal{Z}$ with respect to these fat points are equal to

$$\sum n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(L)}([(\tilde{Z}_i \underset{x'_1}{\times} \mathrm{Spec}(k)) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(L)] \text{ and}$$

$$\sum n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(L)}([(\tilde{Z}_i \underset{x'_2}{\times} \mathrm{Spec}(k)) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(L)]$$

respectively. These cycles coincide according to the condition $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)$. Lemma 1.7.2 shows that $\mathcal{W}_1 = \mathcal{W}_2$.

($2 \Rightarrow 1$): Let $x : \mathrm{Spec}(k) \to S$ be a geometric point of S and let $(x_0, x_1), (y_0, y_1)$ be a pair of fat points over x. From the valuative criterion of properness the fat points have canonical liftings to fat points $(x_0, x_1'), (y_0, y_1')$ of S'. This gives us two geometric points

$$x' = x_1' \circ x_0 : \mathrm{Spec}(k) \to S'$$
$$y' = y_1' \circ y_0 : \mathrm{Spec}(k) \to S'$$

of S' over x. Our statement follows now from the obvious equalities:

$$(x_0, x_1)^*(\mathcal{Z}) = (x_0, x_1')^*(\sum n_i \tilde{Z}_i) = \sum n_i \,\mathrm{cycl}(\tilde{Z}_i \underset{x'}{\times} \mathrm{Spec}(k))$$
$$(x_0, x_1)^*(\mathcal{Z}) = (x_0, x_1')^*(\sum n_i \tilde{Z}_i) = \sum n_i \,\mathrm{cycl}(\tilde{Z}_i \underset{y'}{\times} \mathrm{Spec}(k))$$

□

Example 2.1.22. Let k be any algebraically closed field and let S denote the plane nodal cubic curve $S = \mathrm{Spec}(k[x, y]/(y^2 - x^2 - x^3))$. The blow-up of S in the singular point is the morphism $\pi : \mathbb{A}^1_k \to S$ induced by the ring morphism

$$k[x, y]/(y^2 - x^2 - x^3) \to k[t]$$

given by

$$x \mapsto (t^2 - 1),\ y \mapsto (t^2 - 1)t$$

or in classical coordinates the map is given by $t \mapsto (t^2 - 1, (t^2 - 1)t)$.

Let $p : \mathbb{A}^2_k \to S$ be the composition

$$\mathrm{Spec}(k[t, s]) = \mathbb{A}^2_k \to \mathbb{A}^1_k \xrightarrow{\pi} S$$

where the first map is the projection to the x-axis. In this way we get a scheme $X = \mathbb{A}^2_k \to S$ of finite type over S.

Let $C = V(I)$ be any curve in X such that C projects dominantly onto the x-axis. Then the generic fiber of the morphism $C \to S$ is isomorphic to $\mathrm{Spec}(k(t)[s]/Ik(t)[s])$ which is a zero dimensional scheme. Hence each generic point of an integral curve dominating S has dimension zero in its fiber.

Furthermore letting $s = (x, y) \in S$ denote the singular point of S and let $\gamma : \mathrm{Spec}(k) \to S$ be the corresponding morphism and $\gamma_1, \gamma_2 : \mathrm{Spec}(k) \to \mathbb{A}^1_k$ be the two lifts to $\mathbb{A}^1_k$ and set $L_1 = V(t + 1) \subset \mathbb{A}^2_k$ and $L_2 = V(t - 1) \subset \mathbb{A}^2_k$.

Consider now the diagram where each square is fibered:

$$\begin{tikzcd}
\gamma^*(C) = \gamma_1^*(C)\coprod\gamma_2^*(C) = (L_1\times_{\mathbb{A}_k^2} C\coprod L_2\times_{\mathbb{A}_k^2} C) \arrow[r, hook] \arrow[d, hook] & C \arrow[d, hook] \\
\gamma^*(\mathbb{A}_k^2) = L_1\coprod L_2 = \mathrm{Spec}(k[t,s]/(t+1))\coprod\mathrm{Spec}(k[t,s]/(t-1)) = (\mathbb{A}_k^1\coprod\mathbb{A}_k^1) \arrow[r, hook] \arrow[d] & \mathbb{A}_k^2 \arrow[d] \\
\gamma^*(\mathbb{A}_k^1) = (\mathrm{Spec}(k[t]/(t+1))\coprod\mathrm{Spec}(k[t]/(t-1))) \arrow[r, hook] \arrow[d] & \mathbb{A}_k^1 \arrow[d] \\
\mathrm{Spec}(k(s)) \arrow[r, hook, "\gamma"] & S
\end{tikzcd}$$

Observe that if C is any integral curve dominating $\mathbb{A}_k^1$ then since $\gamma^*(C) = \gamma_1^*(C)\coprod\gamma_2^*(C)$ is an effective Cartier divisor on C ([Stacks, Tag 02OO]), hence the proper transform of C is the graph $C\to\mathbb{A}_k^1\times_S C$ of the morphism $\pi|_C$.

By Proposition 2.1.21 we see that $\mathrm{Cycl}(X/S,0)$ is generated by cycles of the form $\sum a_i z_i - \sum b_j w_j$ where all $a_i, b_j \geq 0$ and all z_i, w_j map to the generic point of S and have dimension zero in their fibre subject to the following requirement: For $l = 1,2$

$$\sum a_i\,\mathrm{cycl}_{L_1\coprod L_2}(L_l\cap Z_i) = \sum b_j\,\mathrm{cycl}_{L_1\coprod L_2}(L_l\cap W_j)$$

where Z_i,W_j denote the closures of z_i, w_j. The monoid $\mathrm{Cycl}^{eff}(X/S,0)$ is generated by cycles of the form z where z is over the generic point of S of dimension zero in its fibre and if Z denotes the closure of z then $Z\cap(L_1\cup L_2)=\emptyset$. An example of an effective relative zero cycle is the generic point of the curve $V(y(x+1)(x-1)+x)^2$.

Consider now the parabola $C_1 = V(t+1-s^2)\subset X$. Observe that C_1 obviously projects densely onto the x-axis and hence dominates S, furthermore we have that

$$\gamma^*(C_1) = \gamma_1^*(C_1)\coprod\gamma_2^*(C_1) = \mathrm{Spec}(k[t,s]/(s^2,t+1))\coprod\mathrm{Spec}(k[t,s]/(2-s^2,t-1)) \cong$$
$$\cong Spec(k[t,s]/(s^2,t+1))\coprod\mathrm{Spec}(k[t,s]/(\sqrt{2}-s,t-1))\coprod\mathrm{Spec}(k[t,s]/(\sqrt{2}+s,t-1)).$$

hence we have

$$\mathrm{cycl}_{\gamma^*(\mathbb{A}_k^2)}(\gamma_1^*(C_1)) = \mathrm{cycl}_{L_1\coprod L_2}(\gamma_1^*(C_1) = 2(s,t+1)$$

while

$$\mathrm{cycl}_{\gamma^*(\mathbb{A}_k^2)}(\gamma_2^*(C_1)) = (\sqrt{2}-s,t-1)+(\sqrt{2}+s,t-1)$$

Thus if we let η_1 denote the generic point of the parabola C_1, then we see from Proposition 2.1.21 that η_1 is not a relative zero cycle.

[2]Lucas Das Dores pointed out this nice example to me.

However let us also consider the lines $C_2 := V(s - \sqrt{2}/2(t+1))$ and $C_3 = V(s + \sqrt{2}/2(t+1))$. We have

$$\begin{aligned}\mathrm{cycl}_{\gamma^*(\mathbb{A}^2_k)}(\gamma_1^*(C_2)) &= (s, t+1)\\ \mathrm{cycl}_{\gamma^*(\mathbb{A}^2_k)}(\gamma_2^*(C_2)) &= (s - \sqrt{2}, (t-1))\end{aligned}$$

and

$$\begin{aligned}\mathrm{cycl}_{\gamma^*(\mathbb{A}^2_k)}(\gamma_1^*(C_3)) &= (s, t+1)\\ \mathrm{cycl}_{\gamma^*(\mathbb{A}^2_k)}(\gamma_2^*(C_3)) &= (s + \sqrt{2}, t-1)\end{aligned}$$

hence letting η_2, η_3 be the generic points of C_2 and C_3 respectively, we see that

$$\mathcal{Z} = \eta_1 - \eta_2 - \eta_3 \in \mathrm{Cycl}(X/S, 0).$$

The following result is insightful and a clear proof can be found in the original source.

Proposition 2.1.23 ([SV00, Proposition 3.1.7]). *Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z} = \sum_{i=1}^k n_i z_i$ be an effective cycle on X which belongs to* $\mathrm{Cycl}(X/S, r)$ *for some $r \geq 0$. Denote by Z_i the closure of the point z_i in X which we consider as an integral closed subscheme in X. Then Z_i is equidimensional of dimension r over S.*

Remark 2.1.24. Proposition 2.1.23 is false for non-effective relative cycles. A counter example is given in [SV00, Ex. 3.1.9].

2.2 Cycles associated with flat subschemes

Let $p : X \to S$ be a morphism of finite type of Noetherian schemes. We denote by $\mathrm{Hilb}(X/S, r)$ (resp. $\mathrm{PropHilb}(X/S, r)$) the set of closed subschemes Z of $X \times_S S$ which are flat (resp. flat and proper) and equidimensional of dimension r over S. Let $\mathbb{N}(\mathrm{Hilb}(X/S, r))$, $\mathbb{N}(\mathrm{PropHilb}(X/S, r))$ (resp. $\mathbb{Z}(\mathrm{Hilb}(X/S, r))$, $\mathbb{Z}(\mathrm{PropHilb}(X/S, r)))$ be the corresponding freely generated abelian monoids (resp. groups). The assignment $S'/S \to \mathbb{N}(\mathrm{Hilb}(X \times_S S'/S', r))$ etc. defines presheaf of abelian monoids (groups) on the category of Noetherian schemes over S. If $\mathcal{Z} = \sum n_i Z_i$ is an element of $\mathbb{Z}(\mathrm{Hilb}(X/S, r))$ and S' is a Noetherian scheme over S we denote by $\mathcal{Z} \times_S S'$ the corresponding element $\sum n_i (Z_i \times_S S')$ of $\mathbb{Z}(\mathrm{Hilb}(X \times_S S'/S', r))$.

In order to define the pullback of relative cycles we will need the following result which is proved in the original source.

Proposition 2.2.1 ([SV00, Proposition 3.2.2]). *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $S' \to S$ be any Noetherian scheme over S. Let further $\mathcal{Z} = \sum n_i Z_i$ be an element of* $\mathbb{Z}(\mathrm{Hilb}(X/S, r))$. *If* $\mathrm{cycl}_X(\mathcal{Z}) = 0$ *then* $\mathrm{cycl}_{X \times_S S'}(\mathcal{Z} \times_S S') = 0$.

It was pointed out by Shane Kelly to the author that Corollary 3.2.4 of [SV00] is incorrect as stated there. The following statement is a correction of loc.cit. suggested by David Rydh.

Corollary 2.2.2. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S which is reduced at its generic points, Z be an element of* $\mathrm{Hilb}(X/S, r)$ *and (x_0, x_1, R) be a fat point over a k-point $x : \mathrm{Spec}(k) \to S$ of S. Then*

$$(x_0, x_1)^*(\mathrm{cycl}_X(Z \underset{S}{\times} S_{\mathrm{red}})) = \mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z \underset{S}{\times} \mathrm{Spec}(k)). \tag{2.2.1}$$

Proof. Using Lemma 2.1.19 we easily reduce the proof to the case where the scheme S is reduced.

Since $Z \times_S \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is flat it follows that if we let η be the image of the generic point under x_1 then we have

$$\mathrm{cycl}(Z \underset{S}{\times} \mathrm{Spec}(R)) = \mathrm{cycl}(Z \underset{S}{\times} \mathrm{Spec}(R_{(0)})) = \mathrm{cycl}(Z_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})). \tag{2.2.2}$$

Furthermore since the projection $pr_{Z_\eta} : \mathrm{Spec}(R_{(0)}) \times_{\mathrm{Spec}(k(\eta))} Z_\eta \to Z_\eta$ is flat it follows from flat-pullback (1.7.2) that

$$\mathrm{cycl}(Z_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})) = pr^*_{Z_\eta}(\mathrm{cycl}(Z_\eta)). \tag{2.2.3}$$

Let now Z_i be the irreducible components of Z, z_i be their generic points and n_i be their multiplicities such that $\mathrm{cycl}_X(Z) = \sum n_i z_i$. We have

$$\sum_{z_i/\eta} n_i z_i = \mathrm{cycl}(Z \underset{S}{\times} \mathrm{Spec}(\mathcal{O}_{S,\eta}))$$

where the sum is taken over those points z_i lying over the point η. Since S is reduced it follows that $\mathcal{O}_{S,\eta} = k(\eta)$ and thus we have

$$\sum_{z_i/\eta} n_i z_i = \mathrm{cycl}(Z_\eta). \tag{2.2.4}$$

Hence by (2.2.3) we have

$$\begin{aligned} \mathrm{cycl}(Z_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})) &= \sum_{z_i/\eta} n_i \,\mathrm{cycl}(((Z_i)_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)}))) \\ &= \sum_i n_i \,\mathrm{cycl}(((Z_i)_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)}))). \end{aligned} \tag{2.2.5}$$

Further since $\phi_{x_1}(Z_i) \to \mathrm{Spec}(R)$ is flat for each i it follows that

$$\begin{aligned} &\mathrm{cycl}(\phi_{x_1}(Z_i)) = \mathrm{cycl}(\phi_{x_1}(Z_i) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(R_{(0)})) = \\ &= \mathrm{cycl}((Z_i \underset{S}{\times} \mathrm{Spec}(R)) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(R_{(0)})) = \\ &= \mathrm{cycl}((Z_i)_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})). \end{aligned}$$

and combining this with (2.2.5) we obtain

$$\mathrm{cycl}(Z_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})) = \sum n_i \,\mathrm{cycl}(\phi_{x_1}(Z_i))$$

and finally (2.2.2) now yields the equality

$$\mathrm{cycl}_{X \times_S \mathrm{Spec}(R)}(Z \underset{S}{\times} \mathrm{Spec}(R)) = \sum n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(R)}(\phi_{x_1}(Z_i)).$$

Proposition 2.2.1 yields now that

$$\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z \underset{S}{\times} \mathrm{Spec}(k)) = \sum n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(\phi_{x_1}(Z_i) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k))$$

and the right hand side is by definition equal to $(x_0, x_1)^*(\mathrm{Cycl}_X(Z))$. □

Example 2.2.3. Let $S = Z = X = \mathrm{Spec}(k[x]/(x^2))$. Let $R = k[t]_{(t)}$ and consider the k-algebra morphism $k[x]/(x)^2 \to R$ given by $x \mapsto 0$. Then this induces a morphism $x_1 : \mathrm{Spec}(R) \to S$, and letting $x_0 : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ being the inclusion of the closed point, we see that we get a fat point (x_0, x_1) of S. Now we have

$$(x_0, x_1)^*(\mathrm{cycl}_X(Z)) = (x_0, x_1)^*(2\,\mathrm{Spec}(k)) = 2(\mathrm{Spec}(k))$$

while on the other hand we have

$$\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z \underset{S}{\times} \mathrm{Spec}(k)) = \mathrm{cycl}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k) = (\mathrm{Spec}(k))$$

hence if S is not reduced at its generic points it is not correct that the cycles $(x_0, x_1)^*(\mathrm{cycl}_X(Z))$ and $\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z \times_S \mathrm{Spec}(k))$ coincide as stated in [SV00, Cor.3.2.4].

Corollary 2.2.4 ([SV00, Corollary 3.2.5])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the image of the map*

$$\mathbb{Z}(\mathrm{Hilb}(X/S, r)) \to \mathrm{Cycl}(X).$$

given by

$$Z \mapsto \mathrm{cycl}_X(Z \underset{S}{\times} S_{\mathrm{red}})$$

lies in $\mathrm{Cycl}_{equi}(X/S, r)$.

2.3 Chow presheaves

Base change

Theorem 2.3.1 ([SV00, Theorem 3.3.1])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $\mathcal{Z}$ be an element of* $\mathrm{Cycl}(X/S, r)$ *and* f :

$T \to S$ be a Noetherian scheme over S. Then there is a unique element $\mathcal{Z}_T \in \mathrm{Cycl}(X \times_S T/T, r) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that for any commutative diagram of the form

$$\begin{tikzcd}
& \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\mathrm{Spec}(k) \arrow[ur, "y_0"] \arrow[dr, "x_0"] & & \\
& \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

where (x_0, x_1) and (y_0, y_1) are fat k-points of S and T respectively one has:

$$(y_0, y_1)^*(\mathcal{Z}_T) = (x_0, x_1)^*(\mathcal{Z}).$$

Proof. We prove the theorem in several steps, starting with the following lemma.

Lemma 2.3.2. *Suppose (y_0, y_1, A) is a fat point over a k-point $y : \mathrm{Spec}(k) \to T$. Then we can find a field extension L/K and a fat point (x_0, x_1, R) over the L-point $\mathrm{Spec}(L) \to S$ such that we have a commutative diagram*

$$\begin{tikzcd}
& \mathrm{Spec}(k) \arrow[r, "y_0"] & \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\mathrm{Spec}(L) \arrow[ur, "\varphi"] \arrow[drr, "x_0"] & & & \\
& & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

Proof. Let $s \in S$ be the (set theoretic) image of the composition $y \circ f : \mathrm{Spec}\, k \to S$. By Corollary 2.1.3 we can find a k'-point $x : \mathrm{Spec}\, k' \to S$ whose image is the point s and a fat point (x_0, x_1, R) over x

It is clear that the scheme $\mathrm{Spec}(k \otimes_{k(s)} k')$ is nonempty, hence we can find a field L together with maps $\varphi : \mathrm{Spec}\, L \to \mathrm{Spec}\, k, \psi : \mathrm{Spec}\, L \to \mathrm{Spec}\, k'$ such that we have a commutative diagram

$$\begin{tikzcd}
& \mathrm{Spec}(k) \arrow[r, "y_0"] & \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\mathrm{Spec}(L) \arrow[ur, "\varphi"] \arrow[dr, "\psi"] & & & \\
& \mathrm{Spec}(k') \arrow[r, "x_0"] & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

∎

Corollary 2.3.3. *Under the notations and assumptions of Theorem 2.3.1 the cycle $\mathcal{Z}_T$ is unique if it exists.*

Proof. Suppose for the sake of contradiction we have two different cycles $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathrm{Cycl}(X \times_S T/T, r)$ which both satisfy the properties of $\mathcal{Z}_T$. By Corollary 2.1.17 there is a k-point $y : \operatorname{Spec} k \to T$ and a fat point (y_0, y_1, A) over y such that

$$(y_0, y_1)^*(\mathcal{Z}_1) \neq (y_0, y_1)^*(\mathcal{Z}_2).$$

By Lemma 2.3.2 we can now find a field extension L/k and a fat point (x_0, x_1, R) over the L-point $\operatorname{Spec} L \to S$ such that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Spec}(k) \arrow[r, "y_0"] & \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\mathrm{Spec}(L) \arrow[ur, "\varphi"] \arrow[drr, "x_0"] & & & \\
 & & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

It follows from Lemma 1.7.2 that

$$(y_0 \circ \varphi, y_1)^*(\mathcal{Z}_1) \neq (y_0 \circ \varphi, y_1)^*(\mathcal{Z}_2)$$

contradicting the assumption that they are both equal to $(x_0, x_1)^*(\mathcal{Z})$. $\square$

We now start proving existence, starting with a small auxiliary lemma.

Lemma 2.3.4. *Let z be a point of X lying over a generic point of S such that z has dimension r in its fiber. Let further Z denote the closure of z. Let $S' \to S_{\mathrm{red}}$ be a blow-up such that the proper transform of Z denoted $\tilde{Z}$ is flat over S' (Theorem 1.2.3). Then $\tilde{Z} \to S'$ is universally equidimensional of dimension r.*

Proof. By Proposition 1.1.23 it is enough to show that $\tilde{Z} \times_{S'} \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ is equidimensional of dimension r for every generic point $\mathrm{Spec}(K) \to S'$. By Lemma 1.2.2 we have that $\tilde{Z} \times_{S'} \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ factors as

$$\tilde{Z} \times_{S'} \mathrm{Spec}(K) \xrightarrow{\cong} Z \times_S \mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(K)$$

The scheme $Z \times_S \mathrm{Spec}(K)$ is either empty in which case it is trivially equidimensional of dimension r or $\mathrm{Spec}(K)$ is $\mathrm{Spec}(k(f(z)))$ in which case $Z \times_S \mathrm{Spec}(K)$ is irreducible of dimension r by Lemma 2.1.13. $\square$

Corollary 2.3.5. *Let S be a Noetherian scheme and $X \to S$ a morphism of finite type over S. Let z be a point of X lying over a generic point of S such that z has dimension r in its fiber. Let further Z denote the closure of z (with* *reduced scheme structure). Suppose that $f : \mathrm{Spec}(R) \to S$ be a morphism from a discrete valuation ring to S mapping the generic point to a generic point of S. Then $\phi_f(Z) \to \mathrm{Spec}(R)$ is universally equidimensional of dimension r.*

Proof. By Theorem 1.2.3 we can find a blow up $S' \to S_{\mathrm{red}}$ satisfying the conditions of Lemma 2.1.8. The desired result now immediately follows from Lemma 2.3.4. $\square$

The key part of the proof (of Theorem 2.3.1) is the simple case when T is a point s of S which follows from the following descriptive lemma. This lemma gives some additional information concerning fat points.

Lemma 2.3.6 ([SV00, Lemma 3.3.2]). *Denote by p the exponential characteristic of the field $k(s)$. Then there exists a unique cycle $\mathcal{Z}_s$ in $\mathrm{Cycl}(X_s, r)[1/p]$ such that for any field extension $k/k(s)$ and any fat point (x_0, x_1) over the k-point $\mathrm{Spec}(k) \to \mathrm{Spec}(k(s)) \to S$ one has $(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_s \otimes_{k(s)} k$.*

Proof. Suppose that $\mathcal{Z} = \sum n_i z_i$ and denote by Z_i the closure of z_i which we consider as an integral closed subscheme of X and choose a blow up $S' \to S_{\mathrm{red}}$ such that the proper transforms $\tilde{Z}_i$ are flat over S'.

Suppose that $k/k(s)$ is a field extension such that the k-point $\mathrm{Spec}(k) \to \mathrm{Spec}(k(s)) \to S$ admits a lifting to S'. By Lemma 2.3.4 we have that $\tilde{Z}_i \times_{S'} \mathrm{Spec}(k)$ is either $\emptyset$ or it is a scheme of pure dimension r. Consider now the cycle

$$\mathcal{Z}_k := \sum n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(\tilde{Z}_i \underset{S'}{\times} \mathrm{Spec}(k)) \in \mathrm{Cycl}(X \underset{S}{\times} \mathrm{Spec}(k), r)$$

By Proposition 2.1.21 and Lemma 1.7.2 it follows that the cycle $\mathcal{Z}_k$ is independent of the choice of lifting of $\mathrm{Spec}(k) \to \mathrm{Spec}(k(s)) \to S$ to S'. Now since the morphism $S' \to S$ is surjective and of finite type we can find a finite normal extension $k_0/k(s)$ such that the point $\mathrm{Spec}\, k_0 \to \mathrm{Spec}\, k(s) \to S$ admits a lifting to S'. By Proposition 2.1.21 and Lemma 1.7.2 again it follows that the cycle $\mathcal{Z}_{k_0}$ is $\mathrm{Gal}(k_0/k(s))$-invariant and hence by Lemma 1.7.3 descends to a cycle $\mathcal{Z}_s \in \mathrm{Cycl}(X_s, r)[1/p]$.

Let now k be any extension of $k(s)$ such that the point $\mathrm{Spec}(k) \to \mathrm{Spec}(k(s)) \to S$ admits a lifting to S' and let L be a composite of k and k_0 over $k(s)$. Then

$$\mathcal{Z}_k \otimes_k L = \mathcal{Z}_{k_0} \otimes_{k_0} L = \mathcal{Z}_s \otimes_{k(s)} L = (\mathcal{Z}_s \otimes_{k(s)} k) \otimes_k L$$

and hence $\mathcal{Z}_k = \mathcal{Z}_s \otimes_{k(s)} k$.

Finally let $k/k(s)$ be a field extension and (x_0, x_1, R) be a fat point over a k-point $\mathrm{Spec}(k) \to \mathrm{Spec}(k(s)) \to S$. By the valuative criterion of properness the morphism $x_1 : \mathrm{Spec}(R) \to S$ has a canonical lifting to S'. This gives us a lifting to S' of our k-point $\operatorname{Spec}(k) \to S$ and it follows from Lemma 2.1.8 and the construction of $\mathcal{Z}_s$ that one has

$$(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_k = \mathcal{Z}_s \otimes_{k(s)} k.$$

$\square$

In the course of the proof of Lemma 2.3.6 we have established that after possibly extending the field $k(s)$ the cycle $\mathcal{Z}_s$ can be computed using flatification:

Lemma 2.3.7 ([SV00, Lemma 3.3.3]). *Let $S' \to S$ be a blow-up such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S' and let $k/k(s)$ be a field extension such that the k-point $\operatorname{Spec}(k) \to \operatorname{Spec}(k(s)) \to S$ admits a lifting to S'. Then $\mathcal{Z}_s \otimes_{k(s)} k = \sum n_i \operatorname{cycl}_{X \times_S \operatorname{Spec}(k)}(\tilde{Z}_i \times_{S'} \operatorname{Spec}(k))$.*

Construction 2.3.8 ([SV00, p.25]). Let $\tau_1, \ldots, \tau_n$ be the generic points of T (recall that T is a Noetherian scheme over S) and $\sigma_1, \ldots, \sigma_n$ be their images in S. For $j = 1, \ldots, n$ let $\mathcal{Z}_{\sigma_j}$ be as in Lemma 2.3.6 and consider the cycles

$$\mathcal{Z}_{\sigma_j} \otimes_{k(\sigma_j)} k(\tau_j) = \sum_l n_{j,l} z_{j,l} \in \operatorname{Cycl}(X \underset{S}{\times} \operatorname{Spec}(k(\tau_j)), r) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Here $n_{j,l}$ are rational numbers and $z_{j,l}$ are points of $X \times_S \operatorname{Spec}(k(\tau_j))$ lying over τ_j and having dimension r in their fibers. Note that $X \times_S \operatorname{Spec}(k(\tau_j))$ is nothing but the fiber of τ_j under the morphism $X \times_S T \to T$ hence the points $z_{j,l}$ may be considered as points of $X \times_S T$. Set $\mathcal{Z}_T := \sum_{j,l} n_{j,l} z_{j,l}$.

We will now show that the cycle $\mathcal{Z}_T$ from Construction 2.3.8 belongs to $\operatorname{Cycl}(X \times_S T/T, r) \otimes_{\mathbb{Z}} \mathbb{Q}$ and has the desired property of Theorem 2.3.1. We give the same proof of this fact as in [SV00].

Consider a commutative diagram of the form

$$\begin{tikzcd}
 & \operatorname{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\operatorname{Spec}(k) \arrow[ur, "y_0"] \arrow[dr, "x_0"'] & & \\
 & \operatorname{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

in which (x_0, x_1) (resp. (y_0, y_1)) is a fat k-point of S (resp. of T). Let as before $S' \to S_{\mathrm{red}}$ denote a blow up of S_{red} such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S'. Lemma 1.3.12 shows that there is a discrete valuation ring A' and a surjective morphism $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ such that the morphism $\operatorname{Spec}(A') \to S$ admits a lifting to S'. Denote the residue field of A (resp. of A')

by k_A (resp. $k_{A'}$) and let k' be a composite of k and $k_{A'}$ over k_A so that we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(A') & \longrightarrow & & & S' \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{Spec}(k) & \xrightarrow{y_0} & \mathrm{Spec}(A) & \xrightarrow{y_1} & T & \longrightarrow & S
\end{array}$$

Letting $Z_{j,l}$ denote the closure of the points $z_{j,l}$ considered as integral subschemes of $X \times_S T$, then by Lemma 2.3.4 and Corollary 2.3.5 we have that the morphisms $\phi_{y_1}(Z_{j,l}) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \to \mathrm{Spec}(A')$ and $\tilde{Z}_i \times_{S'} \mathrm{Spec}(A') \to \mathrm{Spec}(A')$ are flat and universally equidimensional of dimension r. Thus we can consider the following two elements in $\mathbb{Q}(\mathrm{Hilb}(X \times_S \mathrm{Spec}(A')/\mathrm{Spec}(A'), r))$:

$$\mathcal{W} = \sum_i n_i(\tilde{Z}_i \underset{S'}{\times} \mathrm{Spec}(A'))$$
$$\mathcal{W}_1 = \sum_{j,l} n_{j,l}(\phi_{y_1}(Z_{j,l}) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(A'))$$

Assume that y_1 maps the generic point of $\mathrm{Spec}(A)$ to τ_1 , then we further have that

$$\mathcal{W}_1 = \sum_l n_{1,l}(\phi_{y_1}(Z_{1,l}) \underset{\mathrm{Spec}(A)}{\times} \mathrm{Spec}(A'))$$

Lemma 2.3.9 ([SV00, Lemma 3.3.5]).

$$\mathrm{cycl}_{X \times_S \mathrm{Spec}(A')}(\mathcal{W}) = \mathrm{cycl}_{X \times_S \mathrm{Spec}(A')}(\mathcal{W}_1)$$

Proof. Let K (resp. K') denote the quotient field of A (resp. A'). Since the map $p : \mathrm{Spec}(K') \to \mathrm{Spec}(A')$ is flat we have the map $p^* : \mathrm{Cycl}(X \times_S \mathrm{Spec}(A')) \to \mathrm{Cycl}(X \times_S \mathrm{Spec}(K'))$ which we can restrict to relative cycles and extend to rational coefficients and get a map

$$\mathrm{Cycl}(X \underset{S}{\times} \mathrm{Spec}(A')/\mathrm{Spec}(A'), r) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Cycl}(X \underset{S}{\times} \mathrm{Spec}(K')/\mathrm{Spec}(K'), r) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which is clearly injective. Thus we may replace A' by K' everywhere. Furthermore

$$\mathrm{cycl}_{X \times_S \mathrm{Spec}(K')}(\mathcal{W} \underset{\mathrm{Spec}(A')}{\times} \mathrm{Spec}(K')) = \sum n_i \, \mathrm{cycl}_{X \times_S \mathrm{Spec}(K')}(\tilde{Z}_i \underset{S'}{\times} \mathrm{Spec}(K'))$$

and according to Lemma 2.3.7 this cycle is equal to $\mathcal{Z}_{\sigma_1} \otimes_{k(\sigma_1)} K'$.

On the other hand we have:

$$\operatorname{cycl}_{X\times_S \operatorname{Spec}(K')}(\mathcal{W}_1 \underset{\operatorname{Spec} A'}{\times} \operatorname{Spec}(K')) =$$
$$= \sum_l n_{1,l} \operatorname{cycl}_{X\times_S \operatorname{Spec}(K')}([\phi_{y_1}(Z_{1,l}) \underset{\operatorname{Spec}(A)}{\times} \operatorname{Spec}(K)] \underset{\operatorname{Spec}(K)}{\times} \operatorname{Spec}(K')) =$$
$$= \sum_l n_{1,l} \operatorname{cycl}_{X\times_S \operatorname{Spec}(K)}(Z_{1,l} \underset{T}{\times} \operatorname{Spec}(K) \underset{\operatorname{Spec}(K)}{\times} \operatorname{Spec}(K')) =$$
$$= [\sum_l n_{1,l} \operatorname{cycl}_{X\times_S Spec(k(\tau_1))}(Z_{1,l} \underset{T}{\times} \operatorname{Spec}(k(\tau_1)))] \otimes_{k(\tau_1)} K'$$

Furthermore by Lemma 2.1.13 we have that

$$\operatorname{cycl}_{X\times_S \operatorname{Spec}(k(\tau_1))}(Z_{1,l} \underset{T}{\times} \operatorname{Spec}(k(\tau_1))) = z_{1,l}$$

Thus

$$\operatorname{cycl}_{X\times_S \operatorname{Spec}(K')}(\mathcal{W}_1 \underset{\operatorname{Spec} A'}{\times} \operatorname{Spec}(K')) =$$
$$[\sum_l n_{1,l} \operatorname{cycl}_{X\times_S Spec(k(\tau_1))}(Z_{1,l} \underset{T}{\times} \operatorname{Spec}(k(\tau_1)))] \otimes_{k(\tau_1)} K' =$$
$$= (\sum_l n_{1,l} z_{1,l}) \otimes_{k(\tau_1)} K' = (\mathcal{Z}_{\sigma_1} \otimes_{k(\sigma_1)} k(\tau_1)) \otimes_{k(\tau_1)} K' =$$
$$= \mathcal{Z}_{\sigma_1} \otimes_{k(\sigma_1)} K'.$$

which completes the proof of the lemma. $\square$

Proposition 2.2.1 implies now that

$$\operatorname{cycl}_{X\times_S \operatorname{Spec}(k')}(\mathcal{W} \underset{\operatorname{Spec} A'}{\times} \operatorname{Spec}(k')) = \operatorname{cycl}_{X\times_S \operatorname{Spec}(k')}(\mathcal{W}_1 \underset{\operatorname{Spec}(A')}{\times} \operatorname{Spec}(k'))$$

i.e.

$$\sum n_i \operatorname{cycl}_{X\times_S \operatorname{Spec}(k')}(\tilde{Z}_i \underset{S'}{\times} \operatorname{Spec}(k')) = \sum_l n_{1,l} \operatorname{cycl}_{X\times_S \operatorname{Spec}(k')}(\phi_1(Z_{1,l} \underset{\operatorname{Spec}(A')}{\times} \operatorname{Spec}(k')) =$$
$$= (y_0, y_1)^*(\mathcal{Z}_T) \otimes_k k'.$$

On the other hand the cycle $(x_0, x_1)^*(\mathcal{Z}) \otimes_k k'$ coincides with

$$\sum n_i \operatorname{cycl}_{X\times_S \operatorname{Spec}(k')}(\tilde{Z}_i \underset{S'}{\times} \operatorname{Spec}(k'))$$

where this time the morphism $\operatorname{Spec}(k') \to S'$ is a lifting of the same point $\operatorname{Spec}(k') \to \operatorname{Spec}(k) \to \operatorname{Spec}(R) \to S$ obtained using the unique lifting of $\operatorname{Spec}(R) \to S$. Proposition 2.1.21 shows that $(x_0, x_1)^*(\mathcal{Z}) \otimes_k k' = (y_0, y_1)^*(\mathcal{Z}_T) \otimes_k k'$ and hence $(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^*(\mathcal{Z}_T)$. This shows that the cycle $\mathcal{Z}_T$ has the desired property and by Lemma 2.3.2 we also see that $\mathcal{Z}_T \in \operatorname{Cycl}(X \times_S T/T, r)$. Theorem 2.3.1 is proven. $\square$

Notation 2.3.10 ([SV00, p.29])**.** Let $f : T \to S$ be a morphism of Noetherian schemes and $X \to S$ be a scheme of finite type over S. We denote by

$$\mathrm{cycl}(f) : \mathrm{Cycl}(X/S, r) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Cycl}(X \underset{S}{\times} T/T, r) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the homomorphism $\mathrm{cycl}(f)(\mathcal{Z}) = \mathcal{Z}_T$ where $\mathcal{Z}_T$ is defined as in Theorem 2.3.1.

The following remark is similar to the remark on p.27 of [SV00].

Remark 2.3.11. In general it is possible that a cycle with integer coefficients can be pulled back to one whose coefficients are not integers. Such an example due to A.S. Merkurjev will be given in Example 2.5.14.

The statement of the following lemma is stated without proof on p.30 of [SV00].

Lemma 2.3.12. *Suppose we have two morphisms $f : T' \to T$ and $g : T \to S$. Then*

$$\mathrm{cycl}(g \circ f) = \mathrm{cycl}(f) \circ \mathrm{cycl}(g)$$

Proof. Suppose k is a field and A, A' are discrete valuation rings such that we have a commutative diagram of the form:

$$\begin{tikzcd}
 & \mathrm{Spec}(A') \arrow[r, "y_1'"] & T' \arrow[dd, "f"] \\
\mathrm{Spec}(k) \arrow[ur, "y_0'"] \arrow[dr, "y_0"'] & & \\
 & \mathrm{Spec}(A) \arrow[r, "y_1"] & T
\end{tikzcd}$$

By Lemma 2.3.2 we can find a field extension L/k and a discrete valuation ring R such that we get a commutative diagram of the form

$$\begin{tikzcd}
 & & \mathrm{Spec}(A') \arrow[r, "y_1'"] & T' \arrow[dd, "f"] \\
\mathrm{Spec}(L) \arrow[r, "\varphi"] \arrow[ddrr, "x_0"] & \mathrm{Spec}(k) \arrow[ur, "y_0'"] \arrow[dr, "y_0"] & & \\
 & & \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[d, "g"] \\
 & & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

Then

$$(y_0, y_1)^*(\mathrm{cycl}(g)(\mathcal{Z})) \otimes_k L = (y_0 \circ \varphi, y_1)^*(\mathrm{cycl}(g)(\mathcal{Z})) = (x_0, x_1)^*(\mathcal{Z}) =$$
$$= (y_0' \circ \varphi, y_1')^*(\mathrm{cycl}(g \circ f)(\mathcal{Z})) = (y_0', y_1')^*(\mathrm{cycl}(g \circ f)(\mathcal{Z})) \otimes_k L$$

hence

$$(y_0', y_1')^*(\mathrm{cycl}(g \circ f)(\mathcal{Z})) = (y_0, y_1)^*(\mathrm{cycl}(g)(\mathcal{Z}))$$

Thus by uniqueness of $\mathrm{cycl}(f)(\mathrm{cycl}(g)(\mathcal{Z}))$ it follows that

$$(\mathrm{cycl}(g \circ f)(\mathcal{Z})) = \mathrm{cycl}(f)(\mathrm{cycl}(g)(\mathcal{Z})).$$

□

Properties of the pullback homomorphism

Note the easy observations regarding the pullback $\mathrm{cycl}(f)$.

Lemma 2.3.13. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $\mathcal{Z} = \sum a_i z_i \in \mathrm{Cycl}(X/S, r)$.*

(1) *Suppose that $f : T \to S$ is a Noetherian scheme over S. If the set theoretic image $f(T)$ does not intersect the set theoretic image of $\mathrm{Supp}(Z)$ in S, then $\mathrm{cycl}(f) = 0$.*

(2) *If η is a generic point of S and $f : \mathrm{Spec}\, k(\eta) \to S$ is the canonical morphism, then*

$$\begin{aligned}\mathrm{cycl}(f)(\mathcal{Z}) &= \sum a_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}\, k(\eta)}(Z_i \underset{S}{\times} \mathrm{Spec}(k(\eta))) \\ &= \sum_{z_i \textit{ lies over } \eta} a_i z_i \in \mathrm{Cycl}(X \underset{S}{\times} \mathrm{Spec}\, k(\eta), r).\end{aligned}$$

(3) *If $s \in S$ and there exists a fat point (x_0, x_1, R) over the canonical $k(s)$-point $f : \mathrm{Spec}\, k(s) \to S$, then $(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_s$.*

Proof. For (1) note that if we have a field k and fat points $(x_0, x_1, R), (y_0, y_1, A)$ and a commutative diagram of the form

$$\begin{tikzcd}
 & \mathrm{Spec}(A) \arrow[r, "y_1"] & T \arrow[dd, "f"] \\
\mathrm{Spec}(k) \arrow[ur, "y_0"] \arrow[dr, "x_0"'] & & \\
 & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

Then the set theoretic image of $\mathrm{supp}(\mathcal{Z})$ does not contain the image of $x_1 \circ x_0$, hence $(x_0, x_1)^*(\mathcal{Z}) = 0$ thus by uniqueness we deduce that $\mathrm{cycl}(f)(\mathcal{Z}) = 0$.

For (2): For any diagram of the form

$$\begin{tikzcd}
 & \mathrm{Spec}(A) \arrow[r, "y_1"] & \mathrm{Spec}(k(\eta)) \arrow[d, "f"] \\
\mathrm{Spec}(k) \arrow[ur, "y_0"] \arrow[dr, "x_0"'] & & \\
 & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

we easily see that $\mathrm{Spec}(R) \xrightarrow{x_1} S$ factors through $\mathrm{Spec}\, k(\eta) \to S$ thus

$$\begin{aligned}
(x_0, x_1)^* &= \sum a_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z_i \underset{S}{\times} k) = \\
&= (y_0, y_1)^*(\sum a_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}\, k(\eta)}(Z_i \underset{S}{\times} \mathrm{Spec}(k(\eta)))) \\
&= (y_0, y_1)^*(\sum_{z_i \text{ lies over } \eta} a_i z_i).
\end{aligned}$$

For (3): For an independent variable t we have that the canonical inclusion $k(s) \to k(s)[t]_{(t)}$ gives a fat point over the identity map $\mathrm{Spec}\, k(s) \to \mathrm{Spec}\, k(s)$. Consider now the diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(k(s)[t]_{(t)}) \arrow[r, "y_1"] & \mathrm{Spec}(k(s)) \arrow[d, "f"] \\
\mathrm{Spec}(k(s)) \arrow[ur, "y_0"] \arrow[dr, "x_0"'] & & \\
 & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

Then clearly $(y_0, y_1)^*(\mathcal{Z}_s) = \mathcal{Z}_s$ hence $(x_0, x_1)^*(\mathcal{Z}) = \mathcal{Z}_s$.

□

Lemma 2.3.14. *Let S be a Noetherian scheme and $X \to S$ a scheme of finite type over S. Let $\mathcal{Z} = \sum n_i z_i \in \mathrm{Cycl}(X/S, r)$ and let Z_i denote the closure of z_i in X. If $f : S' \to S$ is a blow-up of S_{red} such that the proper transforms $\tilde{Z}_i$ of Z_i are flat over S' then*

$$\mathrm{cycl}(f)(\mathcal{Z}) = \sum n_i \,\mathrm{cycl}_{X \times_S S'}(\tilde{Z}_i)$$

and each $\tilde{Z}_i$ is flat and equidimensional of dimension r over S'.

Proof. Let $\tau_1, \dots, \tau_n$ denote the generic points of S' and let $\sigma_1, \dots, \sigma_n$ be their images in S. By Construction 2.3.8 we have

$$\mathrm{cycl}(f)(\mathcal{Z}) = \sum_{j=1}^{n} \mathcal{Z}_{\sigma_j} \otimes_{k(\sigma_j)} k(\tau_j)$$

where the points here are considered as points in the scheme $X \times_S S'$. By Lemma 2.3.7 we now have that

$$\mathrm{cycl}(f)(\mathcal{Z}) = \sum_{j=1}^{n} \mathcal{Z}_{\sigma_j} \otimes_{k(\sigma_j)} k(\tau_j) =$$

$$= \sum_{j=1}^{n} \sum_{i} n_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k(\tau_j))}(\tilde{Z}_i \underset{S'}{\times} \mathrm{Spec}(k(\tau_j))) = \sum_i n_i \,\mathrm{cycl}_{X \times_S S'}(\tilde{Z}_i).$$

The last statement follows from Lemma 2.3.4 $\square$

Lemma 2.3.15 ([SV00, Lemma 3.3.6])**.** *In the notations and assumptions of Theorem 2.3.1 we have*

$$\mathrm{supp}(\mathcal{Z}_T) \subset (\mathrm{supp}(\mathcal{Z}))_T = \mathrm{supp}(\mathcal{Z}) \underset{S}{\times} T.$$

Proof. The claimed inclusion can be checked fiberwise hence it is sufficient to consider the case $T = \mathrm{Spec}(k)$ for a field k. Using Corollary 2.1.3 we find an extension k'/k and a fat point (x_0, x_1, R) over the k'-point $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \to S$. The defining property of the cycle $\mathcal{Z}_k$ (Lemma 2.3.6) shows that

$$\mathrm{supp}(\mathcal{Z}_k) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(k') = \mathrm{supp}(\mathcal{Z}_{k'}) = \mathrm{supp}((x_0, x_1)^*(\mathcal{Z})) \subset$$

$$\subset \cup_i \phi_{x_1}(Z_i) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k') \subset \cup_i (Z_i \underset{S}{\times} \mathrm{Spec}(R)) \subset \cup_i (Z_i \underset{S}{\times} \mathrm{Spec}(R)) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k') =$$

$$= \mathrm{supp}(\mathcal{Z}) \underset{S}{\times} \mathrm{Spec}(k') = (\mathrm{supp}(\mathcal{Z}) \underset{S}{\times} \mathrm{Spec}(k)) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(k').$$

Since the morphism $X_{k'} \to X_k$ is surjective the above inclusion implies the desired one $\mathrm{supp}(\mathcal{Z}_k) \subset \mathrm{supp}(\mathcal{Z}) \times_S \mathrm{Spec}(k)$. $\square$

Corollary 2.3.16. *Suppose that S is a Noetherian scheme and $\mathcal{Z} \in \mathrm{Cycl}_{equi}(X/S, r)$. Then for any Noetherian scheme T and morphism $T \to S$ we have that the generic points of $\mathrm{Supp}(\mathcal{Z}_T)$ are generic points of $T \times_S \mathrm{Supp}(\mathcal{Z})$.*

Proof. By Lemma 2.3.15 we have the inclusion

$$\mathrm{Supp}(\mathcal{Z}_T) \subset T \underset{S}{\times} \mathrm{Supp}(\mathcal{Z})$$

Let z_i be any generic point of $\mathrm{Supp}(\mathcal{Z}_T)$ and z be a generic point of $T \times_S \mathrm{Supp}(\mathcal{Z})$ whose closure contains the point z_i. Then it is clear that both points lie over a generic point of S say η and by Lemma 1.1.15 it easily follows that we must have $z_i = z$. $\square$

Lemma 2.3.17 ([SV00, Lemma 3.3.7])**.** *Consider a pull-back square of morphisms of finite type of Noetherian schemes of the form*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

and assume that the morphism f is universally open and any generic point of X lies over a generic point of S. Then any generic point of X' lies over a generic point of S'.

Lemma 2.3.18 ([SV00, Lemma 3.3.8])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S, $\mathcal{Z} = \sum n_i z_i$ be an element of* $\mathrm{Cycl}(X/S, r) \otimes \mathbb{Q}$, $f : S' \to S$ *be Noetherian scheme over S and $\mathcal{Z}' = \mathrm{cycl}(f)(\mathcal{Z})$ be the corresponding element of* $\mathrm{Cycl}(X \times_S S'/S', r) \otimes \mathbb{Q}$.

(1) *If f is a universally open morphism then* $\mathrm{supp}(\mathcal{Z}') = (\mathrm{supp}(\mathcal{Z}) \times_S S')_{red}$.

(2) *If f is dominant then* $\mathrm{supp}(\mathcal{Z})$ *is the closure of* $pr_X(\mathrm{Supp}(\mathcal{Z}'))$ *where $pr_X : X \times_S S' \to X$ is the projection.*

Proof. **For (1):** The inclusion

$$\mathrm{supp}(\mathcal{Z}') \subset (\mathrm{supp}(\mathcal{Z}) \underset{S}{\times} S')_{red}$$

follows from Lemma 2.3.15. It is enough to check that these two sets have the same generic points. To this extent note that Lemma 2.3.17 implies immediately that generic points of $\mathrm{supp}(\mathcal{Z}) \times_S S'$ lie over generic points of S'. Hence we may assume that $S' = \mathrm{Spec}(k)$ and the image of S' in S is a generic point η of S. Then k is an extension of $k(\eta)$ and according to Lemma 1.7.2(2) we have

$$\mathrm{supp}(\mathcal{Z}') = \mathrm{supp}(\mathcal{Z}_\eta \otimes_{k(\eta)} k) = (\mathrm{supp}(\mathcal{Z}_\eta) \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(k))_{red}.$$

By Lemma 2.3.13(2) we have

$$\mathcal{Z}_\eta = \sum_{z_i \text{ lies over } \eta} n_i z_i$$

and hence $\mathrm{supp}(\mathcal{Z}_\eta) = \mathrm{supp}(\mathcal{Z}) \times_S \mathrm{Spec}(k(\eta))$.

For (2): It is enough to show that $z_i \in pr_X(\mathrm{Supp}(\mathcal{Z}'))$ for all i. Let η_i denote the image of z_i in S and let η_i' be a point of S' lying over η_i. Note that $\mathcal{Z}_{k(\eta_i')} = (\mathcal{Z}')_{k(\eta_i')}$ and hence by Lemma 2.3.15 we have $\mathrm{supp}(\mathcal{Z}_{k(\eta_i)}) \subset \mathrm{Supp}(\mathcal{Z}')$. Moreover part (1) of this present lemma yields that

$$\mathrm{supp}(\mathcal{Z}_{\eta_i} \underset{\mathrm{Spec}(k(\eta_i))}{\times} \mathrm{Spec}(k(\eta_i')))_{red} = \mathrm{supp}(\mathcal{Z}_{\eta_i'})$$

And since the morphism

$$\mathrm{supp}(\mathcal{Z}_{\eta_i}) \underset{\mathrm{Spec}(k(\eta_i))}{\times} \mathrm{Spec}(k(\eta_i')) \to \mathrm{supp}(\mathcal{Z}_{\eta_i})$$

is surjective and by Lemma 2.3.13(2) we have $z_i \in \mathcal{Z}_{\eta_i}$ the desired result follows. $\square$

The following result is a corrected version of [SV00, Lemma.3.3.10] (see the paragraph preceding Corollary 2.2.2).

Lemma 2.3.19. *Let $T \to S$ be a morphism of Noetherian schemes. Then for any scheme X of finite type over S and $\mathcal{W}$ an element of $\mathbb{Z}(\mathrm{Hilb}(X/S, r))$ we have*

$$[\mathrm{Cycl}_X(\mathcal{W} \underset{S}{\times} S_{\mathrm{red}})]_T = \mathrm{cycl}_{X \times_S T}(\mathcal{W}_T \underset{T}{\times} T_{\mathrm{red}})$$

Proof. This is immediate from Corollary 2.2.2. $\square$

Lemma 2.3.20 ([SV00, Lemma 3.3.12])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S and $f : S' \to S$ be a flat morphism of Noetherian schemes. Assume further that the schemes S and S' are both reduced. Then for any element $\mathcal{Z}$ in $\mathrm{Cycl}(X/S, r)$ one has*

$$\mathrm{cycl}(f)(\mathcal{Z}) = f_X^*(\mathcal{Z})$$

where $f_X = pr_X : X \times_S S' \to X$ and f_X^* is the flat pull-back defined in Section 1.7.

Proof. Let $\tau_1, \ldots, \tau_n$ denote the generic points of S' and we let $\sigma_1, \ldots, \sigma_n$ be their images in S under the morphism f. Since generalizations lift along flat morphisms it follows that all the σ_i are generic points of S. We briefly recall how $\mathrm{cycl}(f)(\mathcal{Z})$ is constructed: Let $n_{i,j} \in \mathbb{Q}$ be such that we have

$$\mathcal{Z}_{\sigma_i} \otimes_{k(\sigma_i)} k(\tau_i) = \sum_j n_{i,j} z_{i,j} \in \mathrm{Cycl}(X \underset{S}{\times} \mathrm{Spec}(k(\tau_i)), r) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Then $\mathrm{cycl}(f)(\mathcal{Z}) = \sum_{i,j} n_{i,j} z_{i,j}$ where $z_{i,j}$ is considered as a point in $X \times_S S'$. By Lemma 2.3.13(2) it follows that if $\mathcal{Z} = \sum a_l z_l$ with Z_i denoting the closure of z_i in X then we have

$$\mathcal{Z}_{\sigma_i} \otimes_{k(\sigma_i)} k(\tau_i) = \sum a_l \, \mathrm{cycl}_{X \times_S \mathrm{Spec}(k(\tau_i))}(Z_l \underset{S}{\times} \mathrm{Spec}(k(\tau_i)))$$

On the other hand we have

$$f_X^*(\mathcal{Z}) = \sum a_l \, \mathrm{cycl}_{X \times_S S'}(Z_l \underset{S}{\times} S')$$

Hence if we can show that the generic points of $Z_l \times_S \operatorname{Spec}(k(\tau_i))$, as i goes through $1, \dots, n$, are exactly the generic points of $S' \times_S Z_l$ then we will have that

$$\mathrm{cycl}_{X \times_S S'}(Z_l \underset{S}{\times} S') = \sum_i \mathrm{cycl}_{X \times_S S'}((Z_l \underset{S}{\times} S') \underset{S'}{\times} \operatorname{Spec}(\mathcal{O}_{S',\tau_i}))$$

Since S' is reduced we have $\mathcal{O}_{S',\tau_i} = k(\tau_i)$ we only need to compare generic points to complete the proof. By continuity of the projection $Z_l \times_S S' \to S'$ we easily see that any generic point of $Z_l \times_S \operatorname{Spec}(k(\tau_i))$ must necessarily be a generic point of $Z_l \times_S S'$. Suppose now that ξ is a generic point of $S' \times_S Z_l$. Since $S' \times_S Z_l \to Z_l$ is flat it follows that ξ is mapped to z_l the generic point of Z_l. Since S is reduced by assumption there is an open dense subset U of S such that $Z_l \to X \to S$ is flat over U, hence we see that the morphism $\operatorname{Spec} k(\xi) \to S' \times_S Z_l$ factors through $S' \times_S Z_l \times_S U$, but the latter is flat over S' hence ξ is mapped to one of the generic points τ_i of S' which completes the proof. □

Chow presheaves

For a scheme X of finite type over a Noetherian scheme S we follow [SV00] and set

$$\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}(S') := \mathrm{Cycl}(X \underset{S}{\times} S'/S', r) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

By Lemma 2.3.12 we then get a presheaf of $\mathbb{Q}$-vector spaces

$$\mathrm{Cycl}(X/S, r)_{\mathbb{Q}} : \mathrm{NoethSch}/S \to \mathrm{Vect}_{\mathbb{Q}}$$

on the category of Noetherian schemes over S. Similarly in the notations of [SV00] we also have the presheaves of $\mathbb{Q}$-vector spaces $\mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}$, $\mathrm{Cycl}_{\mathrm{equi}}(X/S, r)_{\mathbb{Q}}$, and $\mathrm{PropCycl}_{\mathrm{equi}}(X/S, r)_{\mathbb{Q}}$ as well as presheaves of uniquely divisible abelian monoids $\mathrm{Cycl}^{eff}(X/S, r)_{\mathbb{Q}_+}$, $\mathrm{PropCycl}^{eff}(X/S, r)_{\mathbb{Q}_+}$.

Lemma 2.3.21 ([SV00, Lemma 3.3.9]). *Let S be a Noetherian scheme, $X \to S$ be a scheme of finite type over S and $\mathcal{Z}$ be an element of $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$. Then the following conditions are equivalent:*

(1) *For any Noetherian scheme T over S the cycle $\mathcal{Z}_T$ belongs to $\mathrm{Cycl}(X \times_S T/T, r)$ (inparticular $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)$).*

(2) *For any point $s \in S$ the cycle $\mathcal{Z}_s$ belongs to $\mathrm{Cycl}(X_s, r)$.*

(3) *For any point $s \in S$ there exists a separable field extension $k/k(s)$ such that the cycle $\mathcal{Z}_k = \mathcal{Z}_s \otimes_{k(s)} k$ belongs to $\mathrm{Cycl}(X \times_S \operatorname{Spec}(k), r)$.*

Proof. (1) obviously implies (2) which again obviously implies (3). To see that (3) implies (2) apply Lemma 1.7.2 and Lemma 1.4.9.

The final implication $(2 \Rightarrow 1)$ follows from the construction of $\mathcal{Z}_T$. □

Definition 2.3.22 ([SV00, p.30-31])**.** We shall call a relative cycle satisfying the equivalent conditions of Lemma 2.3.21 a relative cycle with *universally integral coefficients*. We denote by $\mathrm{Cycl}(X/S,r)_{UI}$ (resp.

$$\mathrm{PropCycl}(X/S,r)_{UI},\ \mathrm{Cycl}_{equi}(X/S,r)_{UI},\ \mathrm{PropCycl}_{equi}(X/S,r)_{UI}$$

) the subgroup of $\mathrm{Cycl}(X/S)$ (resp. of $\mathrm{PropCycl}(X/S,r)$, $\mathrm{Cycl}_{equi}(X/S,r)$, and of $\mathrm{PropCycl}_{equi}(X/S,r)$) consisting of relative cycles with universally integral coefficients. It is clear that $\mathrm{Cycl}(X/S,r)_{UI}$ (resp. $\ldots$) yields a subpresheaf in the presheaf $\mathrm{Cycl}(X/S,r)_{\mathbb{Q}}$ (resp. $\ldots$). Moreover

$$\mathrm{PropCycl}(X/S,r)_{UI} = \mathrm{Cycl}(X/S,r)_{UI} \cap \mathrm{PropCycl}(X/S,r)_{\mathbb{Q}}$$

etc.

Remark 2.3.23. In [SV00] the presheaves $\mathrm{Cycl}(X/S,r)_{UI}$ and $\mathrm{PropCycl}(X/S,r)_{UI}$ are denoted by the notation $z(X/S,r)$ and $c(X/S,r)$ respectively.

Lemma 2.3.24. *We have a homomorphism of presheaves*

$$\mathrm{cycl} : \mathbb{Z}(\mathrm{Hilb}(X/S,r)) \to \mathrm{Cycl}(X/S,r)_{UI}.$$

given by

$$\mathcal{W} \mapsto \mathrm{cycl}_X(\mathcal{W} \underset{S}{\times} S_{\mathrm{red}})$$

Proof. Follows immediately from Lemma 2.3.19. □

Remark 2.3.25. Lemma 2.3.24 is the corrected statement of Corollary 3.3.11 in [SV00].

Proposition 2.3.26 ([SV00, Proposition 3.3.13])**.** *Let S be a Noetherian scheme of exponential characteristic n and $X \to S$ be a scheme of finite type over S. Then the subgroups* $\mathrm{Cycl}(X \times_S T/T,r)[1/n]$ *in* $\mathrm{Cycl}(X \times_S T/T,r) \otimes_{\mathbb{Z}} \mathbb{Q}$ *for Noetherian schemes T over S form a subpresheaf* $\mathrm{Cycl}(X/S,r)[1/n]$ *in the presheaf* $\mathrm{Cycl}(X/S,r)_{\mathbb{Q}}$.

Proof. Follows easily from Lemma 2.3.6 and the construction of the pullback. □

The next proposition tells us that the definition $\mathrm{Cycl}(X/S,r)_{UI}$ is reasonable. The proof given in the original source is rather clear and therefore omitted here.

Proposition 2.3.27 ([SV00, Proposition 3.3.14])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the quotient presheaf*

$$\mathrm{Cycl}(X/S,r)_{\mathbb{Q}} / \mathrm{Cycl}(X/s,r)_{UI}$$

is a presheaf of torsion abelian groups. i.e. for any $\mathcal{Z} \in \mathrm{Cycl}(X/S,r)$ there is a positive integer N such that $N\mathcal{Z} \in \mathrm{Cycl}(X/S,r)_{UI}$.

Corollary 2.3.28. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the canonical morphisms*

$$\mathrm{Cycl}(X/S, r)_{UI} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Cycl}(X/S, r)_{\mathbb{Q}} \tag{2.3.1}$$

$$\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{UI} \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+} \tag{2.3.2}$$

are isomorphisms, and the analogous statements for the proper and equidimensional versions also hold.

Remark 2.3.29. Suppose that k is a field of exponential characteristic p and S is a Noetherian scheme such that k is a subfield of the residue field $k(s)$ for any point $s \in S$. If $X \to S$ is a scheme of finite type over S and $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)$ is a relative cycle then for any point $s \in S$ it follows from Lemma 2.3.6 that the cycle $\mathcal{Z}_s$ has necessarily coefficients in $\mathbb{Z}[1/p]$. Proposition 2.3.27 tells us that we can find some $N > 0$ such that $N \cdot \mathcal{Z}_s$ has integral coefficients for all $s \in S$ and it is clear that if p^n is the greatest power of p dividing N then $p^n\mathcal{Z} \in \mathrm{Cycl}(X/S, r)_{UI}$.

The last proposition we state in this section tells us that there is a relatively large class of schemes where $\mathrm{Cycl}(X/S, r)$ and $\mathrm{Cycl}(X/S, r)_{UI}$ coincide. Again we refer the reader to the original source for a clear proof.

Proposition 2.3.30 ([SV00, Proposition 3.3.15])**.** *Let S be a regular Noetherian scheme. Then for any scheme of finite type X over S and any $r \geq 0$ one has:*

$$\mathrm{Cycl}(X/S, r) = \mathrm{Cycl}(X/S, r)_{UI}$$
$$\mathrm{Cycl}_{equi}(X/S, r) = \mathrm{Cycl}_{equi}(X/S, r)_{UI}$$

etc.

2.4 Relative cycles over geometrically unibranch schemes

Over geometrically unibranch schemes it turns out that the group of equidimensional relative cycles of dimension r are freely generated by integral closed subschemes which are equidimensional of relative dimension r over the base. In order to prove this we will need the following lemma whose proof can be found in [SV00].

Lemma 2.4.1 ([SV00, Lemma 3.4.1])**.** *Let k be a field, $X \to \mathrm{Spec}(k), S \to \mathrm{Spec}(k)$ be two scheme of finite type over k and Z be a closed subscheme in $X \times_{\mathrm{Spec}(k)} S$ defined by a nilpotent sheaf of ideals which is flat over S. Let further E be an extension of k and s_1, s_2 be two E-points of S over k. If S is geometrically connected then the cycles associated with the closed subschemes $Z \times_{s_1} \mathrm{Spec}(E)$ and $Z \times_{s_2} \mathrm{Spec}(E)$ in $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(E)$ coincide.*

Theorem 2.4.2 ([SV00, Theorem 3.4.2]). *Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Let further $Z \subset X$ be a closed subscheme which is equidimensional of relative dimension r over S. Then* $\mathrm{cycl}_X(Z) \in \mathrm{Cycl}_{equi}(X/S, r)$.

Proof. By Corollary 1.1.19 we may assume that the closed subscheme Z is integral. Choose a blow-up $S' \to S_{\mathrm{red}}$ such that the proper transform $\tilde{Z}$ of Z is flat over S'. Let further k be a field, $s : \mathrm{Spec}(k) \to S$ be a k-point of S and $s_1, s_2 : \mathrm{Spec}(k) \to S'$ be two liftings of s to S'. According to Proposition 2.1.21 we have to show that the cycles $\mathrm{cycl}(\tilde{Z} \times_{s_1} \mathrm{Spec}(k)), \mathrm{cycl}(\tilde{Z} \times_{s_2} \mathrm{Spec}(k))$ coincide. Note that according to Proposition 1.1.20 and Lemma 1.1.25 the closed subscheme $\tilde{Z}$ in $Z \times_S S'$ is defined by a nilpotent sheaf of ideals and hence $\tilde{Z} \times_S \mathrm{Spec}(k)$ is a closed subscheme of $(Z \times_S S') \times_S \mathrm{Spec}(k) = (Z \times_S \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} (S' \times_S \mathrm{Spec}(k))$ defined by a nilpotent sheaf of ideals. The scheme $S' \times_S \mathrm{Spec}(k)$ is geometrically connected according to Proposition C.2.16. Thus our statement follows from Lemma 2.4.1. $\square$

Remark 2.4.3. Theorem 2.4.2 is very similar to [Kol96, p. I.3.17].

Corollary 2.4.4 ([SV00, Corollary 3.4.3]). *Let S be a geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Then the abelian group* $\mathrm{Cycl}_{equi}(X/S, r)$ *(resp.* $\mathrm{PropCycl}_{equi}(X/S, r)$*) is freely generated by cycles of integral closed subschemes Z in X which are equidimensional (resp. proper and equidimensional) of dimension r over S.*

Corollary 2.4.5 ([SV00, Corollary 3.4.4]). *Let S be a Noetherian geometrically unibranch scheme and $X \to S$ be a scheme of finite type over S. Then the abelian group* $\mathrm{Cycl}_{equi}(X/S, r)$ *(resp. the abelian group* $\mathrm{PropCycl}_{equi}(X/S, r)$*) is generated by the abelian monoid* $\mathrm{Cycl}^{eff}(X/S, r)$ *(resp. by the abelian monoid* $\mathrm{PropCycl}^{eff}(X/S, r)$*).*

Proof. Follows immediately from Proposition 2.1.23 and Theorem 2.4.2. $\square$

Corollary 2.4.6 ([SV00, Corollary 3.4.5]). *Let S be a Noetherian regular scheme and X be a scheme of finite type over S. Then the abelian group* $\mathrm{Cycl}_{equi}(X/S, r)_{UI}(S)$ *(resp. the abelian monoid* $\mathrm{Cycl}^{eff}(X/S, r)_{UI}(S)$*) is the free abelian group (resp. free abelian monoid) generated by closed integral subschemes of X which are equidimensional of dimension r over S.*

Proof. Follows immediately from Proposition 2.3.30 and Theorem 2.4.2. $\square$

Corollary 2.4.7 ([SV00, Corollary 3.4.6]). *Let S be a Noetherian regular scheme and X be a scheme of finite type over S. Then the abelian group* $\mathrm{PropCycl}_{equi}(X/S, r)_{UI}(S)$ *(resp. abelian monoid* $\mathrm{PropCycl}^{eff}(X/S, r)(S)$*) is the free abelian group (resp. free abelian monoid) generated by closed integral subschemes of X which are proper and equidimensional of dimension r over S.*

Example 2.4.8. Let S denote the plane nodal cubic curve $S = \mathrm{Spec}(k[x,y]/(y^2 - x^2 - x^3))$. Example C.2.4 shows that S is not unibranch and hence not geometrically unibranch. The aforementioned example also shows that the normalization of S is the morphism $\mathrm{Spec}(k[t]) = \mathbb{A}^1_k \to S$ induced by the ring morphism $x \mapsto (t^2-1),\ y \mapsto (t^2-1)t$. Note that the only point of $\mathbb{A}^1_k/S$ lying over a generic point of S is the generic point ξ of $\mathbb{A}^1_k$ itself. We will show that ξ is not a relative zero cycle, that is $\xi \notin \mathrm{Cycl}(\mathbb{A}^1_k/S, 0)$. To this extent note that the preimage of the singular point of S is the two points $(t+1)$ and $(t-1)$ of $\mathbb{A}^1_k$. This gives us two fat points over the singular point $p : \mathrm{Spec}(k) \to S$ as follows: Consider the ring map $R = k[t]_{(t-1)} \to k$ given by $t \mapsto 1$, this induces a morphism of schemes $x_0 : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ with the closed point as its image. Let further $x_1 : \mathrm{Spec}(R) \to S$ be the obvious composition $\mathrm{Spec}(R) \to \mathbb{A}^1_k \to S$ this gives us a fat point (x_0, x_1, R) over p. Similarly letting $A = k[t]_{(t+1)}$ we get another fat point (y_0, y_1, A) over p.

The maps $id_{\mathrm{Spec}(R)} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ and the canonical map $\mathrm{Spec}(R) \to \mathbb{A}^1_k$ induces a closed embedding

$$\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(R) \times_S \mathbb{A}^1_k.$$

We claim that this closed embedding is in fact equal to $\phi_{x_1}(\mathbb{A}^1_k)$. Indeed it is obviously flat over $\mathrm{Spec}(R)$ and easily seen to be an isomorphism over the generic point of $\mathrm{Spec}(R)$ as well. Now note that

$$\mathrm{Spec}(k) \underset{S}{\times} \mathbb{A}^1_k \cong \mathrm{Spec}(k[t]/(t^2-1)) \cong \mathrm{Spec}(k[t]/(t+1)) \coprod \mathrm{Spec}(k[t]/(t-1))$$

And we have that

$$(x_0, x_1)^*(\xi) = \mathrm{Cycl}_{\mathrm{Spec}(k) \times_S \mathbb{A}^1_k}(\mathrm{Spec}(k) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(R)) = \mathrm{Spec}(k[t]/(t-1))$$

On the other hand we have

$$(y_0, y_1)^*(\xi) = \mathrm{Spec}(k[t]/(t+1))$$

hence $\mathrm{Cycl}(\mathbb{A}^1_k/S, r) = 0$ for all r. We could also have showed this using Proposition 2.1.21.

Example 2.4.9. For simplicity we here fix an algebraically closed field k and consider the plane nodal cubic in the projective plane $S = V_+(y^2z - x^2z - x^3) \subset \mathbb{P}^2_k$. We know that the blow-up of S is $\beta : \mathbb{P}^1_k \to S$ and if we give $\mathbb{P}^1_k$ coordinates λ and μ then the blow up $\beta : \mathbb{P}^1_k \to S$ can be understood in terms of classical points as the map taking $[\lambda : \mu]$ to $[\mu\lambda^2 - \mu^3 : \lambda^3 - \mu^2\lambda : \mu^3]$ and we see that the points $p_1 = (\mu - \lambda), p_2 = (-\mu - \lambda) \in \mathbb{P}^1_k$ corresponding to the coordinates $[1:1]$ and $[-1:1]$ respectively, are both mapped to the singular point (x, y) which corresponds to the coordinates $[0:0:1]$ of S. Now consider a $\mathbb{P}^2_k$ with coordinates X_0, X_1, X_2 and consider the cubic polynomials

$$g(X_0, X_1, X_2) := X_1X_2^2 + X_0X_1^2 \ ; \ h(X_0, X_1, X_2) = X_0X_1X_2 + X_0^2X_2$$

since g and h don't have common factors and involve all three coordinates of $\mathbb{P}^2_k$ we have that

$$X := V(\lambda g + \mu h) \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$$

defines a smooth elliptic surface over $\mathbb{P}^1_k$. We then have a canonical morphism $X \to S$ of finite type. Consider now the curves

$$\begin{aligned} Z_1 :&= V(\mu X_0 - \lambda X_1, X_1 + X_2) \subset X \\ Z_2 :&= V(-\mu X_0 - \lambda X_1, -X_1 + X_2) \subset X. \end{aligned}$$

It is readily checked that the composition of Z_i with the projection onto $\mathbb{P}^1_k$ is an isomorphism, thus letting z_1 and z_2 denote the respective generic points of Z_1 and Z_2, we have that z_1, z_2 are both mapped to the generic point of S. Moreover these points have dimension 0 in their fibers. We will however show that the cycle $\mathcal{Z} = z_1 - z_2 \notin \mathrm{Cycl}(X/S, 0)$.

Let s denote the singular point of S , and consider the corresponding k-point $\gamma : \mathrm{Spec}(k(s)) \to S$. Let R_1 (resp. R_2) be the discrete valuation ring $\mathcal{O}_{\mathbb{P}^1_k,p_1}$ (resp. $\mathcal{O}_{\mathbb{P}^1_k,p_2}$). Then R_1 and R_2 give us fat points over γ

$$\begin{aligned} \mathrm{Spec}\, k \overset{\gamma_0}{\to} \mathrm{Spec}(R_1) \overset{\gamma_1}{\to} S \\ \mathrm{Spec}\, k \overset{\tau_0}{\to} \mathrm{Spec}(R_2) \overset{\tau_1}{\to} S \end{aligned}$$

By either using the fact that Z_1 and Z_2 isomorphic to $\mathbb{P}^1_k$ or the valuative criterion of properness we have that the maps $\mathrm{Spec}(R_i) \to \mathbb{P}^1_k$ both lift to Z_1 and Z_2. It is then readily checked using that $k(S) = k(Z_1) = k(Z_2)$ that for $i = 1, 2$ the graphs of the liftings

$$\mathrm{Spec}(R_1) \hookrightarrow \mathrm{Spec}(R_1) \times_S Z_i$$

satisfy the following

$$(\ \mathrm{Spec}(R_1) \hookrightarrow \mathrm{Spec}(R_1) \times_S Z_i\ ) = \phi_{\gamma_1}(Z_i) \tag{2.4.1}$$

$$(\ \mathrm{Spec}(R_2) \hookrightarrow \mathrm{Spec}(R_1) \times_S Z_i\ ) = \phi_{\tau_1}(Z_i). \tag{2.4.2}$$

Consider the fibers

$$\begin{aligned} X_{p_1} &= X \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_1)) = V(X_1X_2^2 + X_0X_1^2 + X_0X_1X_2 + X_0^2X_2) \\ X_{p_2} &= X \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_2)) = V(-(X_1X_2^2 + X_0X_1^2) + X_0X_1X_2 + X_0^2X_2) \\ (Z_1)_{p_1} &= Z_1 \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_1)) = V(X_0 - X_1, X_1 + X_2) \subset X_{p_1} \\ (Z_1)_{p_2} &= Z_1 \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_2)) = V(X_0 + X_1, X_1 + X_2) \subset X_{p_2} \\ (Z_2)_{p_1} &= Z_2 \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_1)) = V(-X_0 - X_1, -X_1 + X_2) \subset X_{p_1} \\ (Z_2)_{p_2} &= Z_2 \underset{\mathbb{P}^1_k}{\times} \mathrm{Spec}(k(p_2)) = V(-X_0 + X_1, -X_1 + X_2) \subset X_{p_2}. \end{aligned}$$

For $i, j = 1, 2$ consider now the diagram where each square is a pullback square

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(R_j) \arrow[d, hook] & \\
(Z_i)_{p_1} \coprod (Z_i)_{p_2} \cong \mathrm{Spec}(k) \times_S Z_i \arrow[r] \arrow[d, hook] & \mathrm{Spec}\, R_j \times_S Z_i \arrow[r] \arrow[d, hook] & Z_i \arrow[d, hook] \\
X_{p_1} \coprod X_{p_2} \cong \mathrm{Spec}(k) \times_S X \arrow[r] \arrow[d] & \mathrm{Spec}(R_j) \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}\, k(p_1) \coprod \mathrm{Spec}\, k(p_2) \cong \mathrm{Spec}\, k \times_S \mathbb{P}^1_k \arrow[r] \arrow[d] & \mathrm{Spec}(R_j) \times_S \mathbb{P}^1_k \arrow[r] \arrow[d] & \mathbb{P}^1_k \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(R_j) \arrow[r] & S
\end{tikzcd}$$

From which we see that

$(\gamma_0, \gamma_1)*(\mathcal{Z}) = \mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(V(X_0 - X_1, X_1 + X_2) - V(-X_0 - X_1, -X_1 + X_2))$

which we can write out in coordinates as

$$[1 : 1 : -1] - [-1 : 1 : 1]$$

On the other hand we have that

$(\tau_0, \tau_1)^*(\mathcal{Z}) = \mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(V(X_0 + X_1, X_1 + X_2) - V(-X_0 + X_1, -X_1 + X_2))$

which translates to

$$[-1 : 1 : -1] - [1 : 1 : 1]$$

thus $\mathcal{Z} \notin \mathrm{Cycl}(X/S, 0)$.

Example 2.4.10. Let $f : X \to \mathbb{P}^1_k$ be a smooth integral elliptic surface over $\mathbb{P}^1_k$ and let S be as in Example 2.4.9 and let $\beta : \mathbb{P}^1_k \to S$ be the blow up of S in the singular point. Consider the morphism $\beta \circ f : X \to S$. Let η_X denote the generic point of X. We will show that $\eta_X \notin \mathrm{Cycl}^{eff}(X/S, 1)$. To this extent let s be the singular point of S and let $p_1, p_2 \in \mathbb{P}^1_k$ be the two points in the fiber $\beta^{-1}(s)$. The two points p_1, p_2 give liftings $x_1, x_2 : \mathrm{Spec}(k) \to \mathbb{P}^1_k$ of the map $\mathrm{Spec}(k(s)) \to S$. According to Proposition 2.1.21 it is enough to show that if $\tilde{X}$ is the strict transform of X (with respect to the blow-up β) then

$$\mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(\mathrm{Spec}(k) \underset{x_1}{\times} \tilde{X}) \neq \mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(\mathrm{Spec}(k) \underset{x_2}{\times} \tilde{X}).$$

Note that $X_s = (\beta \circ f)^{-1}(s)$ is a disjoint union of the two curves $X_{p_1} = f^{-1}(p_1)$ and $X_{p_2} = f^{-1}(p_2)$ and since any codimension one subscheme of a factorial scheme is an effective Cartier divisor, it follows that X blown up at X_s is just

X itself, but then it follows that the proper transform $\tilde{X}$ of X is the graph of the morphism $X \to \mathbb{P}^1_k$, that is

$$\tilde{X} = \;\; X \lhook\joinrel\not\longrightarrow \mathbb{P}^1_k \times_S X \;\; , \tag{2.4.3}$$

thus we have

$$\operatorname{Spec}(k) \underset{x_1}{\times} \tilde{X} = X_{p_1}$$
$$\operatorname{Spec}(k) \underset{x_2}{\times} \tilde{X} = X_{p_2}$$

proving that $\eta_X \notin \mathrm{Cycl}^{eff}(X/S, 1)$. By exactly the same argument one shows that if C is a smooth curve on X with generic point η_C mapping to the generic point of S, then $\eta_C \notin \mathrm{Cycl}^{eff}(X/S, 0)$.

2.5 Functoriality of Chow presheaves

Proper push forward

The following definition can be found on p.43 in [SV00].

Definition 2.5.1. Let $f : S_1 \to S_2$ be a morphism of Noetherian schemes. We say that a closed subscheme Z of S_1 is proper with respect to f if the restriction of f to Z is a proper morphism. We say that a point s of S_1 is *proper* with respect to f if the closure of s in S_1 which we consider as a reduced closed subscheme is proper with respect to f.

Let S be a Noetherian scheme and $f : X \to Y$ be a morphism of schemes of finite type over S. Let further $\mathcal{Z} = \sum n_i z_i$ be a cycle on X which lies over generic points of S. We say that $\mathcal{Z}$ is proper with respect to f if all the points z_i are proper with respect to f. We define then a cycle $f_*(\mathcal{Z})$ on Y as the sum $\sum n_i m_i f(z_i)$ where m_i is the degree of the field extension $k(z_i)/k(f(z_i))$ if this extension is finite and zero otherwise. Note that if $g : Y \to Z$ is another morphism over S such that the cycle $\sum n_i m_i f(z_i)$ is proper with respect to g then $(g \circ f)_*(\mathcal{Z}) = g_*(f_*(\mathcal{Z}))$.

The proof of the following theorem is somewhat technical and omitted here.

Theorem 2.5.2 ([SV00, Theorem 3.6.1])**.** *Let S be a Noetherian scheme, $p : X_1 \to X_2$ be a morphism of schemes of finite type over S, and $f : S' \to S$ be a Noetherian scheme over S. Set $X_i' := X_i \times_S S'$ $(i = 1, 2)$ and denote by $p' : X_1' \to X_2'$ the corresponding morphism over S'. Let further $\mathcal{Z} = \sum n_i Z_i$ (resp. $\mathcal{W} = \sum m_j W_j$) be an element of $\mathbb{Z}(\mathrm{Hilb}(X_1/S, r))$ (resp. of $\mathbb{Z}(\mathrm{Hilb}(X_2/S, r))$). Assume that the closed subschemes Z_i are proper with respect to p and*

$$p_*(\mathrm{cycl}_{X_1}(\mathcal{Z}) = \mathrm{cycl}_{X_2}(\mathcal{W}).$$

Then the cycle $\mathrm{cycl}_{X_1'}(\mathcal{Z} \times_S S')$ *is proper with respect to* p' *and we have*

$$p'_*(\mathrm{cycl}_{X_1'}(\mathcal{Z} \underset{S}{\times} S')) = \mathrm{cycl}_{X_2'}(\mathcal{W} \underset{S}{\times} S').$$

Lemma 2.5.3. *Let* S *be a Noetherian scheme and* $X \to S$ *be a finite type morphism. Let* Z *be an integral closed subscheme of* X *and let* $x : \mathrm{Spec}(R) \to S$ *be a morphism from the spectrum of a discrete valuation ring. Let* $i : \mathrm{Spec}(R_{(0)}) \times_S X \to \mathrm{Spec}(R) \times_S X$ *be the canonical projection. Then*

$$i_*(\mathrm{cycl}_{\mathrm{Spec}(R_{(0)}) \times_S X}(\mathrm{Spec}(R_{(0)}) \underset{S}{\times} Z) = \mathrm{cycl}_{\mathrm{Spec}(R) \times_S X}(\phi_x(Z))$$

Proof. Since $\phi_x(Z) \to \mathrm{Spec}(R) \times_S Z$ is an isomorphism over the generic point of $\mathrm{Spec}(R)$ it follows that the morphism i induces a one to one correspondence between the generic points of $\mathrm{Spec}(R_{(0)}) \times_S Z$ and those of $\phi_x(Z)$. The result now follows from the general fact that for any scheme T over a scheme V we have that if v is a point of V then the stalks of the scheme $\mathrm{Spec}(\mathcal{O}_{V,v}) \times_V T$ are isomorphic to stalks of T at points mapping to generalizations of v. $\square$

Lemma 2.5.4. *Let* $p : X \to Y$ *be a morphism of schemes of finite type over a Noetherian scheme* S. *Suppose that* $\mathcal{Z} = \sum n_i z_i \in \mathrm{Cycl}(X/S, r)$ *is a cycle such that the points* z_i *are proper with respect to* p *and let* Z_i *(resp.* W_i*) denote the closure of* z_i *(resp.* $p(z_i)$*) considered as an integral closed subscheme of* X *(resp. of* Y*). Suppose that* $x_1 : \mathrm{Spec}(R) \to S$ *is a morphism from the spectrum of a discrete valuation ring mapping the generic point to a generic point of* S.

(1) *We have that* $\phi_{x_1}(Z_i)$ *is flat and equidimensional of relative dimension* r *for all* i *hence*

$$\mathcal{Z}_0 := \sum n_i \phi_{x_1}(Z_i) \in \mathbb{Z}(\mathrm{Hilb}(X \underset{S}{\times} \mathrm{Spec}(R)/\mathrm{Spec}(R), r)).$$

(2) *Set* $m_i = [k(z_i) : k(p(z_i))]$ *if this field extension is finite and zero otherwise, we have that for each* i *such that* $m_i \neq 0$ *the morphism* $\phi_{x_1}(W_i) \to \mathrm{Spec}(R)$ *is flat and equidimensional of relative dimension* r, *thus*

$$\mathcal{W}_0 := \sum n_i m_i \phi_{x_1}(W_i) \in \mathbb{Z}(\mathrm{Hilb}(Y \underset{S}{\times} \mathrm{Spec}(R)/\mathrm{Spec}(R), r)).$$

(3) *We have the equality*

$$\mathrm{cycl}(\mathcal{W}_0) = (p \underset{S}{\times} \mathrm{Spec}(R))_*(\mathrm{cycl}(\mathcal{Z}_0)).$$

Proof. The first statement is Corollary 2.3.5. To prove (2) it is by Proposition 1.1.23 enough to check that the generic fiber of $\phi_{x_1}(W_i) \to \mathrm{Spec}(R)$ is of pure dimension r or empty. To this extent let η_i be the generic point of S which z_i lies over. Then we have a map of integral $k(\eta_i)$-varieties $(Z_i) \times_S \operatorname{Spec} k(\eta_i) \to (W_i) \times_S \operatorname{Spec} k(\eta_i)$ and we have

$$\begin{aligned}
\dim(Z_i \underset{S}{\times} \operatorname{Spec}(k(\eta_i))) &= \operatorname{tr.deg}_{k(\eta_i)} k(z_i) \\
\dim(W_i \underset{S}{\times} \operatorname{Spec}(k(\eta_i))) &= \operatorname{tr.deg}_{k(\eta_i)} k(p(z_i))
\end{aligned}$$

from which it follows that if $\dim(W_i \times_S \operatorname{Spec}(k(\eta_i))) < r$ then $m_i = 0$ and since relative dimension is stable under base change (2) now follows.

We now prove the final statement. Consider the cycle $\mathcal{W} = \sum n_i m_i p(z_i)$. From Lemma 2.5.3 and functoriality of push forward of cycles it is enough to show that

$$(p \underset{S}{\times} \operatorname{Spec}(R_{(0)}))_*(\operatorname{cycl}(\sum n_i Z_i \underset{S}{\times} \operatorname{Spec}(R_{(0)}))) = \operatorname{cycl}_{Y \times_S \operatorname{Spec}(R_{(0)})}(\sum n_i m_i W_i \underset{S}{\times} \operatorname{Spec}(R_{(0)})).$$

To this extent let U be an open dense subscheme of S_{red} such that $Z_i \times_S U \to U, W_i \times_S U \to U$ are flat for all i. We can then consider the relative cycle $\mathcal{Z}_U \in \operatorname{Cycl}(X \times_S U/U, r)$ which we may view as an element of $\mathbb{Z}(\operatorname{Hilb}(X \times_S U/U, r))$. Note that $p_*(\mathcal{Z}_U) = \mathcal{W}_U$ hence by Theorem 2.5.2 it follows that

$$\begin{aligned}
&(p \underset{S}{\times} \operatorname{Spec}(R_{(0)}))_*(\operatorname{cycl}(\sum n_i Z_i \underset{S}{\times} \operatorname{Spec}(R_{(0)})) = (p \underset{S}{\times} \operatorname{Spec}(R_{(0)})_*(\operatorname{cycl}_{X \times_S \operatorname{Spec}(R_{(0)})}(\mathcal{Z}_U \underset{U}{\times} \operatorname{Spec}(R \\
&= \operatorname{cycl}_{Y \times_S \operatorname{Spec} R_{(0)}}(\mathcal{W}_U \underset{U}{\times} \operatorname{Spec}(R_{(0)})) = \operatorname{cycl}_{Y \times_S \operatorname{Spec}(R_{(0)})}(\sum n_i m_i W_i \underset{S}{\times} \operatorname{Spec}(R_{(0)})).
\end{aligned}$$

□

Proposition 2.5.5 ([SV00, Proposition 3.6.2])**.** *Let $p : X \to Y$ be a morphism of schemes of finite type over a Noetherian scheme S and $\mathcal{Z} = \sum n_i z_i$ be an element of* $\operatorname{Cycl}(X/S, r)$ *such that the points z_i are proper with respect to p. Then the following statements hold:*

(1) *The cycle $p_*(\mathcal{Z})$ on Y belongs to* $\operatorname{Cycl}(Y/S, r)$. *Explicitly we have that if (x_0, x_1, R) is a fat point over a k-point of S then*

$$(x_0, x_1)^*(p_*(\mathcal{Z})) = (p \underset{S}{\times} \operatorname{Spec}(k))_*((x_0, x_1)^*(\mathcal{Z})).$$

(2) *For any morphism $f : S' \to S$ of Noetherian schemes the cycle* $\operatorname{cycl}(f)(\mathcal{Z})$ *has the form $\sum m_j z'_j$ where the points z'_j are proper with respect to $p' = p \times_S S'$ and moreover*

$$p'_*(\operatorname{cycl}(f)(\mathcal{Z})) = \operatorname{cycl}(f)(p_*(\mathcal{Z})).$$

Proof. Our proof expands slightly on the one found in loc.cit. Let k be a field, $x : \operatorname{Spec}(k) \to S$ be a k-point of S and (x_0, x_1, R) be a fat point of S over x.

Denote by Z_i (resp. W_i) the closure of z_i (resp. $p(z_i)$) considered as integral closed subschemes of X (resp. of Y). Let $\mathcal{Z}_0, \mathcal{W}_0$ be as in Lemma 2.5.4. It follows then from this same lemma that $\mathrm{cycl}(\mathcal{W}_0) = (p \times_S \mathrm{Spec}(R))_*(\mathrm{cycl}(\mathcal{Z}_0))$. Theorem 2.5.2 implies now that

$$(x_0, x_1)^*(p_*(\mathcal{Z})) = \mathrm{cycl}(\mathcal{W}_0 \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k)) =$$
$$= (p \underset{S}{\times} \mathrm{Spec}(k))_*(\mathrm{cycl}(\mathcal{Z}_0 \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k)) = (p \underset{S}{\times} \mathrm{Spec}(k))_*((x_0, x_1)^*(\mathcal{Z})).$$

Thus the cycle $(x_0, x_1)^*(p_*(\mathcal{Z}))$ is independent of the choice of fat point (x_0, x_1, R) over x proving (1).

For (2): We first note from Lemma 2.3.15 that the points z'_j are proper with respect to p'. Let further k be a field and let $y : \mathrm{Spec}(k) \to S'$ be a k-point of S' and $x = f \circ y$. Suppose that (y_0, y_1, A) is a fat point of S' over y and (x_0, x_1, R) a fat point of S over x. We then have from part (1) that

$$\begin{aligned}
(y_0, y_1)^* p'_*(\mathrm{cycl}(f)(\mathcal{Z})) &= (p' \underset{S'}{\times} \mathrm{Spec}(k))_*((y_0, y_1)^*(\mathrm{cycl}(f)(\mathcal{Z})) \\
&= (p \underset{S}{\times} \mathrm{Spec}(k))_*((y_0, y_1)^*(\mathrm{cycl}(f)(\mathcal{Z})) \\
&= (p \underset{S}{\times} \mathrm{Spec}(k))_*((x_0, x_1)^*(\mathcal{Z})) \\
&= (x_0, x_1)^*(p_*(\mathcal{Z})).
\end{aligned}$$

proving that $p'_*(\mathrm{cycl}(f)(\mathcal{Z}))$ satisfies the property defining the cycle $\mathrm{cycl}(f)(p_*(\mathcal{Z}))$. □

Corollary 2.5.6 ([SV00, Corollary 3.6.3])**.** *Let S be a Noetherian scheme and $f : X \to Y$ be a morphism of schemes of finite type over S. Then there are homomorphisms:*

$$\begin{aligned}
f_* &: \mathrm{PropCycl}(X/S, r)_{UI} \to \mathrm{PropCycl}(Y/S, r)_{UI} \\
f_* &: \mathrm{PropCycl}_{equi}(X/S, r)_{UI} \to \mathrm{PropCycl}_{equi}(Y/S, r)_{UI} \\
f_* &: \mathrm{PropCycl}^{eff}(X/S, r)_{UI} \to \mathrm{PropCycl}^{eff}(Y/S, r)_{UI}.
\end{aligned}$$

If moreover f is proper then we also have homomorphisms

$$\begin{aligned}
f_* &: \mathrm{Cycl}(X/S, r)_{UI} \to \mathrm{Cycl}(Y/S, r)_{UI} \\
f_* &: \mathrm{Cycl}_{equi}(X/S, r)_{UI} \to \mathrm{Cycl}_{equi}(Y/S, r)_{UI} \\
f_* &: \mathrm{Cycl}^{eff}(X/S, r)_{UI} \to \mathrm{Cycl}^{eff}(Y/S, r)_{UI}.
\end{aligned}$$

such that for any composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of schemes of finite type over S one has $(g \circ f)_* = g_* \circ f_*$.

Flat pullback

Recall the flat pullback of algebraic cycles from Section 1.7. We will show that this also behaves well with Chow presheaves.

The following result is stated without proof in [SV00].

Lemma 2.5.7 ([SV00, Lemma 3.6.4]). *Let S be a Noetherian scheme and $f : X \to Y$ be a flat equidimensional morphism of relative dimension n of schemes of finite type over S. Then for any $\mathcal{Z} \in \mathrm{Cycl}(Y/S, r) \otimes \mathbb{Q}$ one has $f^*(\mathcal{Z}) \in \mathrm{Cycl}(X/S, r+n) \otimes \mathbb{Q}$. More precisely we have that for any fat point (x_0, x_1) over a k-point of S we have*

$$(f \underset{S}{\times} \mathrm{Spec}(k))^*((x_0, x_1)^*(\mathcal{Z})) = (x_0, x_1)^*(f^*(\mathcal{Z}))$$

and hence for any Noetherian scheme $g : S' \to S$ we have

$$\mathrm{cycl}(g)(f^*(\mathcal{Z})) = (f \underset{S}{\times} S')^*(\mathrm{cycl}(g)(\mathcal{Z})).$$

Proof. Let Z be an integral closed subscheme of Y and let T denote the scheme theoretic image of Z in S and η the generic point of T. Note that since the closed embedding $T \to S$ is a monomorphism and $\mathrm{Spec}(k(\eta)) \to S$ factors through T we have $Z \times_S \mathrm{Spec}(k(\eta)) = Z \times_T \mathrm{Spec}(k(\eta))$ and $(f^{-1}(Z))_\eta = f^{-1}(Z) \times_S \mathrm{Spec}(k(\eta)) = f^{-1}(Z) \times_T \mathrm{Spec}(k(\eta))$. From this and flatness we easily see that $\mathrm{cycl}(f^{-1}(Z)) = \sum n_j w_j = \mathrm{cycl}((f^{-1}(Z))_\eta)$. Also note that if (x_0, x_1, R) is a fat point over a k-point of S such that x_1 maps the generic point of $\mathrm{Spec}(R)$ to η then if W_j denotes the closure of w_j in Z then we have

$$\mathrm{cycl}((W_j)_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})) = \mathrm{cycl}(\phi_{x_1}(W_j))\ ; \tag{2.5.1}$$

$$\mathrm{cycl}((Z)_\eta \underset{\mathrm{Spec}(k(\eta))}{\times} \mathrm{Spec}(R_{(0)})) = \mathrm{cycl}(\phi_{x_1}(Z)). \tag{2.5.2}$$

Now using Lemma 1.7.2 we easily see that

$$\mathrm{cycl}(\sum n_j \phi_{x_1}(W_j)) = \mathrm{cycl}(\phi_{x_1}(Z) \underset{Z \times_S \mathrm{Spec}(R)}{\times} (f^{-1}(Z) \underset{S}{\times} \mathrm{Spec}(R))).$$

Since

$$\sum n_j \phi_{x_1}(W_j), \phi_{x_1}(Z) \underset{Z \times_S \mathrm{Spec}(R)}{\times} (f^{-1}(Z) \underset{S}{\times} \mathrm{Spec}(R)))$$

is an element of

$$\mathbb{Z}(\mathrm{Hilb}(f^{-1}(Z) \underset{S}{\times} \mathrm{Spec}(R)/\mathrm{Spec}(R), r+n))$$

it now follows from Theorem 2.5.2 that

$$\begin{aligned}&\sum n_j \,\mathrm{cycl}(\phi_{x_1}(W_j) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k)) \\ &= \mathrm{cycl}(\phi_{x_1}(Z) \underset{Z \times_S \mathrm{Spec}(R)}{\times} (f^{-1}(Z) \underset{S}{\times} \mathrm{Spec}(R)) \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(k))\end{aligned}$$

which by Lemma 1.7.2 is equal to

$$(f \underset{S}{\times} \operatorname{Spec}(k))^*(\operatorname{cycl}(\phi_{x_1}(Z) \underset{S}{\times} \operatorname{Spec}(k)))$$

which now by linearity proves the claim that

$$(f \underset{S}{\times} \operatorname{Spec}(k))^*((x_0, x_1)^*(\mathcal{Z})) = (x_0, x_1)^*(f^*(\mathcal{Z})).$$

This immediately shows that $f^*(\mathcal{Z}) \in \operatorname{Cycl}(X/S, r+n) \otimes \mathbb{Q}$ and the final assertion is easily proved using what we have already showed to check that $(f \times_S S')^*(\operatorname{cycl}(g)(\mathcal{Z}))$ satisfies the defining property of $\operatorname{cycl}(g)(f^*(\mathcal{Z}))$. $\square$

Construction 2.5.8 ([SV00, p.47]). Let S be a Noetherian scheme, $f : X \to Y$ be a flat (resp. flat and proper) equidimensional morphism of relative dimension n of schemes of finite type over S and $F(-,-)$ be one of the presheaves $\operatorname{Cycl}(-,-)_{UI}$, $\operatorname{Cycl}^{eff}(-,-)_{UI}$ and $\operatorname{Cycl}_{equi}(-,-)_{UI}$ (resp. one of the presheaves of proper relative cycles: $\operatorname{PropCycl}(-,-)_{UI}$,$\operatorname{PropCycl}^{eff}(-,-)_{UI}$ and $\operatorname{PropCycl}_{equi}(-,-)_{UI}$). If $\mathcal{Z}$ is a cycle on Y which belongs to $F(Y/S, r)$ then by Lemma 2.5.7 the cycle $f^*(\mathcal{Z})$ belongs to $F(X/S, r+n)$ and this construction gives us homomorphisms of presheaves

$$f^* : F(Y/S, r) \to F(X/S, r+n).$$

For any composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of flat (resp. flat and proper) equidimensional morphisms of schemes of finite type over S we have $(g \circ f)^* = f^* \circ g^*$.

Proposition 2.5.9 ([SV00, Proposition 3.6.5]). *Let S be a Noetherian scheme. Consider a pull-back square of schemes of finite type over S of the form:*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

such that the morphism f is flat and equidimensional of dimension d. Assume further that either f is also proper and $F(-,-)$ is one of the presheaves $\operatorname{PropCycl}(-,-)_{UI}$, $\operatorname{PropCycl}^{eff}(-,-)_{UI}$, $\operatorname{PropCycl}_{eff}(-,-)_{UI}$ *or that p is proper and $F(-,-)$ is one of the presheaves* $\operatorname{Cycl}(-,-)_{UI}$, $\operatorname{Cycl}^{eff}(-,-)_{UI}$, $\operatorname{Cycl}_{equi}(-,-)_{UI}$ *. Then the following diagram of presheaves commutes:*

$$\begin{array}{ccc} F(Y/S, n) & \xrightarrow{g^*} & F(Y'/S, n+d) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} \\ F(X/S, n) & \xrightarrow{f^*} & F(X'/S, n+d) \end{array}$$

Proof. It follows immediately from the definitions and Proposition 1.7.9. $\square$

Comparison with the construction given in [SV96]

The paper [SV96] also introduces a notion of a relative zero cycle. There the base S is always taken to be a normal algebraic scheme. In this subsection we will apply the push-forward homomorphisms to show that the pullback of proper equidimensional zero cycles over normal schemes in the sense of [SV00] coincides with the pullback constructed in [SV96].

Let us first show that our definition of proper equidimensional relative cycles of dimension 0 coincides with the definition of relative zero cycles in op.cit.

Lemma 2.5.10. *Let S be a normal integral Noetherian scheme and $X \to S$ be any scheme of finite type over S. Then the group* $\mathrm{PropCycl}_{equi}(X/S, 0)$ *is freely generated by integral closed subschemes of X which are finite and surjective over S. In particular if X, S are schemes of finite type over a field k of exponential characteristic p then in the notation of Section 6 of [SV96] we have*

$$\mathrm{z}_0^c(X)(S) = \mathrm{PropCycl}_{equi}(X/\operatorname{Spec}(k))(S)[1/p]$$

Proof. Since normal schemes are geometrically unibranched Corollary 2.4.4 tells us that $\mathrm{PropCycl}^{eff}(X/S, 0)$ is the monoid generated by cycles of integral closed subschemes of X which are proper and equidimensional of relative dimension 0. The desired result now follows from [Stacks, Tag 02LS]. □

We will shortly show that for finite cycles over normal algebraic schemes our base change homomorphisms coincide with those used in [SV96], but we now first follow [SV00] and introduce a generalization of the latter construction to an arbitrary normal Noetherian scheme. For an integral scheme X we denote by $k(X)$ its field of functions.

Definition 2.5.11 ([SV00, Def.3.6.6])**.** A finite surjective morphism $f : Y \to S$ of integral Noetherian schemes is called a *pseudo-Galois covering* if the field extension $k(Y)/k(S)$ is normal and the canonical homomorphism

$$\operatorname*{Aut}_S(Y) \to \operatorname*{Aut}_{k(S)}(k(Y)) = \mathrm{Gal}(k(Y)/k(S)) \tag{2.5.3}$$

is an isomorphism.

Notation 2.5.12 ([SV00, p.48])**.** Let S be a normal integral Noetherian scheme, X be an integral scheme and $p : X \to S$ be a finite surjective morphism and assume [3] that there exists a pseudo-Galois covering $f : Y \to S$ and an S-morphism $q : Y \to X$. Let $g : S' \to S$ be any Noetherian integral scheme over S. Denote by X_i' the irreducible components of $X' = X \times_S S'$ and by x_i' (resp. x) the generic point of X_i' (resp. of X). Theorem 2.4.2

[3] It is stated in [SV00] that such a covering always exists. For a Nagata scheme S we can use a normalization.

tells us that $x \in \mathrm{Cycl}_{equi}(X/S,0)$ and [SV96, Lemma 5.11] shows that each point x'_i lies over the generic point of S' hence the cycle $\mathrm{cycl}(g)(x)$ is of the form $\mathrm{cycl}(g)(x) = \sum n_i x'_i$ with $n_i \in \mathbb{Q}$. Let G be the Galois group $\mathrm{Gal}(k(Y)/k(S)) = \mathrm{Aut}_S(Y)$. Denote by Y'_j the irreducible components of $Y' = Y \times_S S'$. [SV96, Corollary 5.10] tells us that G permutes the components Y'_j transitively so that in particular the field extensions $k(Y'_j)/k(S')$ are all isomorphic. Denote by $l(i)$ the number of components Y'_j lying over X'_i and by l the total number of components Y'_j.

Proposition 2.5.13 ([SV00, Proposition 3.6.7]). *Under the assumptions and notation from Notation 2.5.12 one has:*

$$n_i = \frac{[k(X):k(S)]l(i)}{[k(X'_i):k(S')]l}.$$

Proof. We give the same proof as loc.cit. Denote the generic point of Y (resp. of Y'_j) by y (resp. by y'_j). The cycle y is in $\mathrm{Cycl}(Y/S,0)$ by Theorem 2.4.2 and has the following obvious properties:

1. $f_*(y) = [k(Y):k(S)]s$ where s is the generic point of S.
2. $q_*(y) = [k(Y):k(X)]x$.
3. $\sigma_*(y) = y$ for any $\sigma \in G$.

Consider the cycle $\mathrm{cycl}(g)(y) = \sum m_j y'_j$. Proposition 2.5.5 shows that

$$(f \underset{S}{\times} S')_*(\mathrm{cycl}(g)(y)) = [k(Y):k(S)]s'$$

where s' is the generic point of S' thus

$$\sum m_j[k(Y'_j):k(S')] = [k(Y):k(S)].$$

Moreover for any $\sigma \in G$ we have

$$(\sigma \underset{S}{\times} S'))*(\mathrm{cycl}(g)(y)) = \mathrm{cycl}(g)(\sigma_*(y)) = \mathrm{cycl}(g)(y).$$

Since the action of G on the set $y'_1, \ldots, y'_l$ is transitive we conclude that all multiplicities m_j are the same and equal to $\frac{[k(Y):k(S)]}{l[k(Y'_j):k(S')]}$. Finally

$$\mathrm{cycl}(g)(x) = \frac{1}{[k(Y):k(X)]}(q \underset{S}{\times} S')_*(\mathrm{cycl}(g)(y))$$

and hence

$$n_i = \frac{1}{[k(Y):k(X)]}\sum_{y'_j/x'_i} m_j[k(Y'_j):k(X'_i)] = \frac{[k(X):k(S)]l(i)}{[k(X'_i):k(S')]l}.$$

□

The following example shows that the pullback of a relative cycle with integral coefficients may not have integral coefficients.

Example 2.5.14 (Merkurjev)**.** Assume that char $k > 0$ and let $a, b \in k^\times$ be two elements independent modulo $(k^\times)^p$. Set $A := k[T_0, T_1, T_2]/(aT_0^p + bT_1^p - T_2^p)$, $S = \mathrm{Spec}(A)$. One verifies that A is an integrally closed domain so that S is a normal integral scheme. Let X be the normalization of S in the field $k(S)(\gamma)$, where $\gamma^p = b/a$. One can then check that $X = \mathrm{Spec}(k(\alpha, \beta)[T_1, T_2])$, where α, β is a p'th root of a, b respectively, and the image of T_0 in $k(\alpha, \beta)[T_1, T_2]$ is $\alpha^{-1}T_2 - \gamma T_1$. Note that clearly the map $X \overset{id_X}{\to} X \to S$ is a pseudo-Galois covering. Set finally $S' = \mathrm{Spec}(k)$ and let $S' \to S$ be the only singular point of S, that is the point $(T_0, T_1, T_2) \in \mathrm{Spec}(A) = S$. The scheme $X' = \mathrm{Spec}(k(\alpha, \beta))$ is irreducible and the multiplicity of the only component of X' is given by the formula

$$n = \frac{[k(X) : k(S)]}{[k(X') : k(S')]} = p/p^2 = 1/p. \tag{2.5.4}$$

Remark 2.5.15. Essentially the same example, but using different arguments to compute the multiplicities in the base change, is given in [SV00, Example 3.5.10(1)].

Rational equivalence of algebraic cycles

Arguably the most central notion appearing in intersection theory is that of rational equivalence of algebraic cycles. This notion can both be defined in terms of orders of vanishing of rational functions or by algebraic cycles parametrized by the projective line (see [Ful98, Ch.1, Sec.3,Sec.6] respectively). The latter of these formulations can be (re)stated in the language of relative cycles as follows:

Definition 2.5.16. Let $X \to \mathrm{Spec}(k)$ be a scheme of finite type over a field k. An r-dimensional cycle $\mathcal{Z} \in \mathrm{Cycl}(X, r)$ is *rationally equivalent* to zero if there exists a relative effective cycle $\mathcal{W} \in \mathrm{Cycl}^{eff}(X/\mathrm{Spec}(k), r)(\mathbb{P}^1_k)$ such that

$$\mathcal{Z} = \mathrm{cycl}(t_0)(\mathcal{W}) - \mathrm{cycl}(t_\infty)(\mathcal{W}), \tag{2.5.5}$$

where $t_0 : \mathrm{Spec}(k) \to \mathbb{P}^1_k$ and $t_\infty : \mathrm{Spec}(k) \to \mathbb{P}^1_k$ denote the usual zero and infinity points of $\mathbb{P}^1_k$.

Cycles rationally equivalent to zero on an algebraic scheme X clearly form a subgroup $\mathrm{Rat}(X, r)$ of $\mathrm{Cycl}(X, r)$. The *Chow group* of r cycles on X is defined to be the group $\mathrm{CH}(X, r) := \mathrm{Cycl}(X, r)/\mathrm{Rat}(X, r)$. Two cycles $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathrm{Cycl}(X, r)$ are *rationally equivalent* if $\mathcal{Z}_1 - \mathcal{Z}_2 \in \mathrm{Rat}(X, r)$.

Proposition 2.5.17. *Let $f : X \to Y$ be a morphism of schemes of finite type over a field k. Then the following statements hold true:*

1. *If f is proper then*

$$f_*(\mathrm{Rat}(X,r)) \subset \mathrm{Rat}(Y,r) \tag{2.5.6}$$

2. *If f is flat and equidimensional of dimension d then*

$$f^*(\mathrm{Rat}(Y,r)) \subset \mathrm{Rat}(X,r+d) \tag{2.5.7}$$

Proof. The two statements follow easily from the definitions together with Corollary 2.5.6 and Lemma 2.5.7. □

2.6 Relation with Kollár's families of cycles

Apart from the Chow presheaves we have already seen due to Suslin and Voevodsky there are also other theories of presheaves whose sections on $\mathrm{Spec}(k)$ are algebraic cycles on an algebraic scheme over k. In this part we will show that the theory due to Kollár developed in [Kol96, Ch. I, Sec. 3 and 4] is strongly related to the theory of relative cycles and chow presheaves which we have studied earlier in this text.

Since a lot of the notions defined in [SV00] and [Kol96, Ch. I, Sec. 3 and 4] are very similar, but use different notation we shall save the reader from having to grasp more notation and mostly stick with the notation and definitions already given in this text and instead explain how the definitions in [Kol96] differ from those in [SV00].

Well defined families of proper algebraic cycles

Due to Corollary 2.1.7 we can give the following formulation of [Kol96, Ch. 1, Def. 3.9]

Definition 2.6.1. Let $g : Z \to S$ be a proper morphism of schemes over S with Z an integral scheme and S a reduced Noetherian scheme. By generic flatness there is an open subscheme $U \subset S$ such that Z is flat over U. Let $f : \mathrm{Spec}(R) \to S$ be a morphism from the spectrum of a discrete valuation ring such that the image of the generic point is contained in U and let s denote the image of the closed point. Let $\phi_f(Z)$ denote the closed subscheme of $Z \times_S \mathrm{Spec}(R)$ satisfying the properties of Corollary 2.1.7 and let $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ be the closed point. The *cycle theoretic fiber* of g at s along f denoted by $\lim_{f\to s}(Z/S)$ is defined to be the cycle

$$\lim_{f\to s}(Z/S) := \mathrm{cycl}_{\mathrm{Spec}(k)\times_{\mathrm{Spec}(k(s))} Z}(\mathrm{Spec}(k) \underset{\mathrm{Spec}(R)}{\times} \phi_f(Z))$$

Remark 2.6.2. This definition is essentially the same as the pullback along a fat point. The only difference is a weaker requirement on where f maps the generic point of the spectrum of the discrete valuation ring and that we pullback $\phi_f(Z)$ to the closed point of $\mathrm{Spec}(R)$ and not to some given field mapping to the closed point of $\mathrm{Spec}(R)$.

Definition 2.6.3 ([Kol96, Ch. I, Def.3.8]). Let X be an algebraic scheme over a field k. Let L and K be two field extensions of k. Two cycles

$$\mathcal{Z} \in \mathrm{Cycl}(X_L),\ \mathcal{W} \in \mathrm{Cycl}(X_K)$$

are said to be *essentially the same cycles* if for every field F and embeddings $L \to F,\ K \to F$ we have

$$\mathcal{Z} \otimes_L F = \mathcal{W} \otimes_K F.$$

If $\mathcal{Z}$ is essentially the same as $\mathcal{W}$ we shall denote this by

$$\mathcal{Z} \stackrel{ess}{=} \mathcal{W}$$

Remark 2.6.4. Note that $\stackrel{ess}{=}$ is an equivalence relation.

The following definition is standard.

Definition 2.6.5. Suppose that X is an algebraic scheme over a field k and we have field extensions $k \subset F \subset K$. If $\mathcal{Z}$ is an r-cycle on X_K we say that $\mathcal{Z}$ is *defined over* F if there is a cycle $\mathcal{Z}_F \in \mathrm{Cycl}(X_F, r)$ such that

$$(\mathcal{Z}_F) \otimes_F K = \mathcal{Z}$$

Definition 2.6.6. Let S be a reduced Noetherian scheme and $X \to S$ a morphism of finite type. A *well defined familiy of r-dimensional proper algebraic cycles of* X/S is a cycle $\mathcal{Z} = \sum a_i z_i$ on X satisfying the following conditions

(1) The morphism $\mathrm{Supp}(\mathcal{Z}) \to S$ is proper and equidimensional of dimension r. In particular letting Z_i denote the closure of the points z_i (considered as integral subschemes of X) there is a largest open dense subset U of S such that all the Z_i are flat over U.

(2) For every point $s \in S$ letting $k(s)^{Perf}$ denote the perfect closure of the residue field there exists a cycle

$$\mathcal{Z}_\infty[s] \in \mathrm{Cycl}((\mathrm{Supp}(\mathcal{Z}) \underset{S}{\times} \mathrm{Spec}(k(s))) \underset{\mathrm{Spec}(k(s))}{\times} \mathrm{Spec}(k(s)^{Perf}), r) \subset$$
$$\subset \mathrm{Cycl}(X_s \underset{\mathrm{Spec}(k(s))}{\times} \mathrm{Spec}(k(s)^{Perf}), r)$$

satisfying the following property :

For every morphism $f : \mathrm{Spec}(R) \to S$ from the spectrum of a discrete valuation ring R mapping the generic point to a point in U and the closed point to s we have

$$\mathcal{Z}_\infty[s] \stackrel{ess}{=} \sum a_i \lim_{f \to s} (Z_i/S).$$

The next two results of ours tell us that this definition coincides with the definition of proper equidimensional relative cycles of dimension r given in [SV00, Definition 3.1.3] (2.1.11).

Lemma 2.6.7. *Let S be a Noetherian scheme and consider a relative cycle $\mathcal{Z} = \sum a_i z_i \in \mathrm{Cycl}(X/S, r)$ so $a_i \in \mathbb{Z}$. Let $s \in S$ and let $k(s)^{Perf}$ denote the perfect closure of the residue field of s. Let $\mathcal{Z}_s$ be the cycle constructed in Lemma 2.3.6. Then the cycle*

$$\mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf} \in \mathrm{Cycl}(X_s \underset{\mathrm{Spec}(k(s))}{\times} \mathrm{Spec}(k(s)^{Perf})/\,\mathrm{Spec}(k(s)^{Perf}), r)_{\mathbb{Q}}$$

is an element of $\mathrm{Cycl}(X_s \times_{\mathrm{Spec}(k(s))} \mathrm{Spec}(k(s)^{Perf}), r)$.

Proof. Let Z_i denote the closures of z_i with induced reduced subscheme structure and let $S' \to S$ be a blow-up of S_{red} such that the proper transforms $\tilde{Z}_i$ are all flat over S' (Theorem 1.2.3). Since $S' \to S$ is a surjective morphism of finite type we can find a finite normal field extension L of $k(s)$ such that the map $\mathrm{Spec}(L) \to \mathrm{Spec}(k(s)) \to S$ admits a lifting to S'. From Lemma 2.3.7 it follows that $\mathcal{Z}_s \otimes_{k(s)} L$ has integral coefficients. Furthermore since the field extension $L/k(s)$ is finite normal it follows from Lemma 1.4.21 that the extension $L^{\mathrm{Gal}(L/k(s))}/k(s)$ is purely inseparable and the extension $L/L^{\mathrm{Gal}(L/k(s))}$ is separable. From part (3) of Lemma 2.3.21 it immediately follows that $\mathcal{Z}_s \otimes_{k(s)} L^{\mathrm{Gal}(L/k(s))}$ has integral coefficients and since $\mathrm{Spec}(k(s)^{Perf}) \to \mathrm{Spec}(k(s))$ factors through $\mathrm{Spec}(L^{\mathrm{Gal}(L/k(s))})$ we are done. $\square$

Proposition 2.6.8. *Let S be a reduced Noetherian scheme and $X \to S$ a morphism of finite type. Then a cycle $\mathcal{Z} = \sum a_i z_i$ on X is a well defined family of proper r-dimensional algebraic cycles of X/S in the sense of Definition 2.6.6 if and only if $\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)$.*

Proof. Suppose $\mathcal{Z}$ is a well-defined family of proper r-dimensional algebraic cycles of X/S. Then it follows by definition that $\mathrm{Supp}(\mathcal{Z})$ is proper and equidimensional of relative dimension r over S. Furthermore if $x : \mathrm{Spec}(k) \to S$ is any k-point of S with image $s \in S$ and $(x_0, x_1), (y_0, y_1)$ are any two fat points over x then we clearly have

$$(x_0, x_1)^*(\mathcal{Z}) \overset{ess}{=} \mathcal{Z}_\infty[s] \overset{ess}{=} (y_0, y_1)^*(\mathcal{Z})$$

thus

$$(x_0, x_1)^*(\mathcal{Z}) = (y_0, y_1)^* \mathcal{Z}.$$

Conversely suppose that $\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)$ then part ((1)) of Definition 2.6.6 is satisfied by definition. For any point $s \in S$ we claim that the cycle $\mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf}$ satisfies the property of $\mathcal{Z}_\infty[s]$. Note first of all that by Lemma 2.6.7 and Lemma 2.3.15 it follows that

$$\mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf} \in \mathrm{Cycl}((\mathrm{Supp}(\mathcal{Z}) \underset{S}{\times} \mathrm{Spec}(k(s))) \underset{\mathrm{Spec}(k(s))}{\times} \mathrm{Spec}(k(s)^{Perf}), r).$$

Furthermore if Z_i denotes the closure of z_i considered as an integral subscheme of X for each i and U is a dense open subset of S such that all the Z_i are flat over U and let $S' \to S$ be a blow-up of S with center disjoint from U such that the proper transforms $\tilde{Z}_i$ are flat over S' (Theorem 1.2.3) then if $f : \mathrm{Spec}(R) \to S$ is a morphism from the spectrum of a discrete valuation ring taking the generic point to a point in U and the closed point to a point $s \in S$, then Lemma 2.1.8 immediately yields a lifting $\mathrm{Spec}(R) \to S'$ and letting $x : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ denote the closed point of the scheme $\mathrm{Spec}(R)$

$$\sum a_i \lim_{f \to s}(Z_i/S) = \sum a_i \,\mathrm{cycl}_{X \times_{\mathrm{Spec}(k(s))} k}(\mathrm{Spec}(k) \underset{S'}{\times} \tilde{Z}_i)$$

and so by Lemma 2.3.7 we obtain

$$\sum a_i \lim_{f \to s}(Z_i/S) = \mathcal{Z}_s \otimes_{k(s)} k$$

which is essentially the same as $\mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf}$. This completes the proof. □

In the course of the proof of Proposition 2.6.8 we established the following:

Remark 2.6.9. Let S be a reduced Noetherian scheme and $X \to S$ a morphism of finite type and $\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)$. Then $\mathcal{Z}$ is a well-defined family of proper r-dimensional algebraic cycles of X/S and for each point $s \in S$ we have

$$\mathcal{Z}_\infty[s] = \mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf}.$$

Definition 2.6.10. (See [Kol96, Ch. I, Sec. 4 , Def. 4.7]) Let $X \to S$ be a finite type morphism over a reduced Noetherian scheme S and let $\mathcal{Z} = \sum a_i z_i$ a well defined family of proper r-dimensional algebraic cycles of X/S. We say that $\mathcal{Z}$ satisfies the *field of definition condition* if for every point $s \in S$ the cycle $\mathcal{Z}_\infty[s]$ is defined over $k(s)$.

Proposition 2.6.11. *Under the assumptions and notations of Definition 2.6.10 we have that $\mathcal{Z}$ satisfies the field of definition condition if and only if*

$$\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)_{UI}$$

Proof. From Proposition 2.6.8 and Remark 2.6.9 we have that $\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)$ and that

$$\mathcal{Z}_\infty[s] = \mathcal{Z}_s \otimes_{k(s)} k(s)^{Perf}$$

for each $s \in S$ which is defined over $k(s)$ for every $s \in S$ if and only if $\mathcal{Z} \in \mathrm{PropCycl}_{equi}(X/S, r)_{UI}$. □

2.7 Loci of vanishing and effectiveness

In this section we prove that the locus where a given relative cycle is effective/vanishes is a closed subset of S. As far as we know this result cannot be found in the literature.

Lemma 2.7.1. *Let S be a reduced Noetherian scheme and $f : X \to S$ a morphism of finite type. Consider two elements $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{N}(\mathrm{Hilb}(X/S, r))$. Let $\mathcal{V}_1, \mathcal{V}_2$ be two effective cycles on X such that*

$$\mathcal{V}_1 - \mathcal{V}_2 = \mathrm{cycl}_X(\mathcal{W}_1) - \mathrm{cycl}_X(\mathcal{W}_2)$$

Let further Z be the union of those irreducible components occuring in both $\mathrm{cycl}_X(\mathcal{W}_1)$ and in $\mathrm{cycl}_X(\mathcal{W}_2)$ with the same multiplicities and set

$$U := \mathrm{Supp}(\mathcal{W}_2) \setminus (\mathrm{Supp}(\mathcal{V}_1) \cup Z).$$

If $\mathcal{V}_2 \neq 0$ then the following assertions are true:

(1) *The set U is a non-empty open subset of $\mathrm{Supp}(\mathcal{W}_2)$.*

(2) *If $s \in f(U)$ then the cycle $\mathrm{cycl}_{X_s}(\mathcal{W}_1 \times_S \mathrm{Spec}(k(s))) - \mathrm{cycl}_{X_s}(\mathcal{W}_2 \times_S \mathrm{Spec}(k(s)))$ is not effective. Equivalently if we set $\mathcal{Z}_1 = \mathrm{cycl}_X(\mathcal{W}_1), \mathcal{Z}_2 = \mathrm{cycl}_X(\mathcal{W}_2)$ then the relative cycle $(\mathcal{Z}_1)_s - (\mathcal{Z}_2)_s = (\mathcal{Z}_1 - \mathcal{Z}_2)_s$ (see Lemma 2.3.19) is not effective.*

(3) *The set $f(U)$ is an open subset of S.*

Proof. It is clear that U is an open subset of $\mathrm{Supp}(\mathcal{W}_2)$ and since $\mathcal{V}_2 \neq 0$ and $\mathrm{Supp}(\mathrm{cycl}_X(\mathcal{W}_2))$ is equidimensional of dimension r over S it follows that $\mathrm{Supp}(\mathcal{W}_2) \nsubseteq \mathrm{Supp}(\mathcal{V}_1) \cup Z$ thus (1) follows. For (2) consider any element $u \in U$ and set $s = f(u) \in f(U) \subset S$. Note that since $u \notin \mathrm{Supp}(\mathcal{V}_1) \cup Z$ it follows that $u \in \mathrm{Supp}(\mathcal{V}_2)$. Let $\mathcal{V}_1 = \mathrm{cycl}(\sum a_i Z_i)$ and $\mathcal{V}_2 = \mathrm{cycl}(\sum b_j Z_j)$ where Z_i, Z_j are integral schemes and let $S' \to S$ be a blow-up of S such that the proper transforms $\tilde{Z}_i, \tilde{Z}_j$ are flat over S' for all i, j. Let j be such that $u \in Z_j$ then since $\tilde{Z}_j \to Z_j$ is surjective we can find some $\tilde{u} \in \tilde{Z}_j$ mapping to $u \in Z_j$ and let s' be the image of $\tilde{u}$ in S'. By Lemma 2.3.7 we have that

$$(\mathcal{V}_1 - \mathcal{V}_2)_s \otimes_{k(s)} k(s') =$$
$$\sum a_i \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k(s'))}(\tilde{Z}_i \underset{S'}{\times} \mathrm{Spec}(k(s'))) - \sum b_j \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k(s'))}(\tilde{Z}_j \underset{S'}{\times} \mathrm{Spec}(k(s'))).$$

Note that since u is not in the image of any of the maps $\tilde{Z}_i \times_{S'} \mathrm{Spec}(k(s')) \to (Z_i)_s$ but $\tilde{u} \in \tilde{Z}_j \times_{S'} \mathrm{Spec}(k(s'))$ for some j it follows that $(\mathcal{V}_1 - \mathcal{V}_2)_s \otimes_{k(s)} k(s')$ cannot be effective and since the pullback of an effective relative cycle must necessarily be effective it follows that $(\mathcal{Z}_1)_s - (\mathcal{Z}_2)_s$ is not effective which completes the proof of (2). The last assertion follows easily from [Stacks, Tag 0CVT]. $\square$

Corollary 2.7.2. *Let S be a reduced Noetherian scheme and $f : X \to S$ a morphism of finite type. Consider two elements $\mathcal{W}_1, \mathcal{W}_2 \in \mathbb{N}(Hilb(X/S, r))$. The set of points $s \in S$ such that*

$$\mathrm{cycl}_{X_s}(\mathcal{W}_1 \underset{S}{\times} \mathrm{Spec}(k(s))) - \mathrm{cycl}_{X_s}(\mathcal{W}_2 \underset{S}{\times} \mathrm{Spec}(k(s)))$$

is an effective cycle on X_s is a closed subset of S.

Proof. By Lemma 2.3.19 it is enough to show that the set

$$\mathrm{Eff}_S(\mathcal{Z}) = \{s \in S \mid \mathcal{Z}_s \text{ is an effective cycle on } X_s\},$$

where $\mathcal{Z} := \mathrm{cycl}_X(\mathcal{W}_1) - \mathrm{cycl}_X(\mathcal{W}_2) \in \mathrm{Cycl}(X/S, r)$, is a closed subset of S.

If $\mathcal{Z}$ is not already an effective cycle on X it follows from Lemma 2.7.1 that there is a closed subset $S_1 \subsetneq S$ where $\mathrm{Eff}_S(\mathcal{Z}) \subset S_1$. We can consider the set S_1 as a reduced closed subscheme of S and we denote the closed embedding $i_1 : S_1 \to S$. Then we can consider the pullback

$$\mathrm{cycl}(i_1)(\mathcal{Z}) = \mathrm{cycl}_{S_1 \times_S X}(\mathcal{W}_1 \underset{S}{\times} S_1) - \mathrm{cycl}_{S_1 \times_S X}(\mathcal{W}_2 \underset{S}{\times} S_1);$$

if this is not an effective cycle we can repeat the procedure and we obtain a descending chain of closed subschemes

$$S \supsetneq S_1 \supsetneq S_2 \supsetneq \dots.$$

Since S is a Noetherian topological space it follows that the aforementioned sequence must stabilize at some S_N which means that

$$\mathrm{cycl}_{X \times_S S_N}(\mathcal{W}_1) - \mathrm{cycl}_{X \times_S S_N}(\mathcal{W}_2)$$

is an effective cycle and $\mathrm{Eff}_S(\mathcal{Z}) = S_N$. $\square$

Recall from Lemma 2.3.13 that if $p : S' \to S_{\mathrm{red}} \to S$ is a blow-up of S_{red} making the proper transforms $\tilde{Z}_i$ flat over S' then one has that

$$\mathrm{cycl}(p)(\mathcal{Z}) = \sum a_i \,\mathrm{cycl}(\tilde{Z}_i).$$

This allows us to prove the last few results of this section.

Proposition 2.7.3. *Let S be a Noetherian scheme and $f : X \to S$ a scheme of finite type over S. Let F be either of the presheaves $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}, \mathrm{Cycl}(X/S, r)_{UI}$ and let F^{eff} denote its respective effective counterpart. For a relative cycle $\mathcal{Z} = \sum a_i z_i \in F(S)$ the set*

$$\mathrm{Eff}_S(\mathcal{Z}) = \{s \in S \mid \mathcal{Z}_s \in F^{eff}(\mathrm{Spec}(k(s)))\}$$

is a closed subset of S.

Proof. Let Z_i be the closure of the point z_i and let $S' \to S_{\text{red}} \to S$ be a blow-up of S_{red} such that the proper transforms $\tilde{Z}_i$ are flat over S'. As $\text{cycl}(p)(\mathcal{Z}) = \sum a_i \,\text{cycl}(\tilde{Z}_i) \in \mathbb{Z}(\text{Hilb}(X \times_S S'/S', r))$ it follows immediately from Corollary 2.7.2 that $\text{Eff}_{S'}(\text{cycl}(p)(\mathcal{Z}))$ is a closed subset of S'. Since p is proper it is therefore enough to prove that

$$\text{Eff}_S(\mathcal{Z}) = p(\text{Eff}_{S'}(\text{cycl}(p)(\mathcal{Z}))).$$

It follows from functoriality that if $s \in \text{Eff}_S(\mathcal{Z})$ then for any point $s' \in S'$ lying over s we must have that $(\text{cycl}(p)(\mathcal{Z}))_{s'} \in F^{eff}(\text{Spec}(k(s')))$ thus $\text{Eff}_S(\mathcal{Z}) \subset p(\text{Eff}_{S'}(\text{cycl}(p)(\mathcal{Z})))$. Conversely if $s \notin \text{Eff}_S(\mathcal{Z})$ then if $s' \in S'$ is any point lying over s then from functoriality and Lemma 2.3.19 we have

$$(\text{cycl}(p)(\mathcal{Z}))_{s'} = \mathcal{Z}_s \otimes_{k(s)} k(s')$$

and it is clear that the right hand side is not an effective cycle, thus $s \notin p(\text{Eff}_{S'}(\text{cycl}(p)(\mathcal{Z})))$ which completes the proof. $\square$

Proposition 2.7.4. *Let S be a Noetherian scheme and $f : X \to S$ a scheme of finite type over S. Let F be either of the presheaves $\text{Cycl}(X/S, r)_{\mathbb{Q}}, \text{Cycl}(X/S, r)_{UI}$. For a relative cycle $\mathcal{Z} = \sum a_i z_i \in F(S)$ the set*

$$\text{Null}_S(\mathcal{Z}) = \{s \in S \mid \mathcal{Z}_s = 0 \in F(\text{Spec}(k(s)))\}$$

is a closed subset of S.

Proof. Note that $\text{Null}_S(\mathcal{Z}) = \text{Eff}_S(\mathcal{Z}) \cap \text{Eff}_S(-\mathcal{Z})$ and apply Proposition 2.7.3. $\square$

2.8 An overview of the literature

Since this chapter has a lot of overlap with [SV00] we have provided a table explaining how our presentation of the material taken from op.cit. (and other sources) compares to the original.

Comparison table			
Statement	Reference(s)	Statement comparison	Proof
Lemma 2.1.2	[GD61, (7.1.7)]	Translation	Translation
Lemma 2.1.6	[GD67, (2.8.5)]	Translation	Sketch
Proposition 2.1.18	[SV00, Cor.3.1.6]	Identical	Different
Proposition 2.1.21	[SV00, Prop.3.1.5]	Identical	Identical
Corollary 2.2.2	[SV00, Cor.3.2.4]	Correction	Corrects and expands
Theorem 2.3.1	[SV00, Thm.3.3.1]	Identical	Occasionally expands
Lemma 2.3.12	[SV00, p.30]	Identical	Added
Lemma 2.3.15	[SV00, Lem.3.3.6]	Identical	Identical
Lemma 2.3.18	[SV00, Lem.3.3.8]	Identical	Slight expansion
Lemma 2.3.19	[SV00, Lem.3.3.10]	Corrected	
Lemma 2.3.20	[SV00, Lem.3.3.12]	Identical	Added
Lemma 2.3.21	[SV00, Lem.3.3.9]	Identical	Slightly different
Lemma 2.3.24	[SV00, Cor.3.3.11]	Corrected	
Theorem 2.4.2	[SV00, Thm.3.4.2]	Identical	Identical
Proposition 2.5.5	[SV00, Prop.3.6.2]	Identical	Slight expansion
Lemma 2.5.7	[SV00, Lem.3.6.4]	Identical	Added
Proposition 2.5.13	[SV00, Prop.3.6.7]	Identical	Identical
Example 2.5.14	[SV96, p.77]	Identical	Identical

Chapter 3

The h-topologies

A fundamental tool in the study of relative cycles is the h-topology, a Grothendieck topology on the category of schemes with more coverings than what one encounters in other areas of algebraic geometry. This means that a presheaf has to allow its sections to be glued together in rather many ways for it to be a sheaf in the h-topology. As one might expect being a sheaf in such a fine topology is a potential obstruction to representability by a scheme. On the other hand if one has a sheaf in the h-topology one then has a nice set of tools to prove the existence of sections with desirable properties.

The layout of this chapter is as follows: First we introduce the h and qfh topologies and discuss their basic properties closely following [Voe96]. Notably we recall Voevodsky's proof that an h-covering of a Noetherian Nagata scheme can be refined to a Zariski covering followed by a proper surjective map. As an application of the limit methods from the first chapter we show that the Nagata hypthesis is not necessary[1]. We then follow this up by giving some consequences of the refinement result. We also briefly discuss a few cousins of the h-topology where one has more control over field arithmetic, before moving on to studying presheaves of relative cycles in the context of the h-topologies essentially following [SV00].

Finally in the last section we recall from [SV96] how sheaves in the qfh-topology interact with quotients by finite groups.

Throughout this chapter every scheme is separated.

3.1 The h topology

(Universal) Topological epimorphisms

Definition 3.1.1 ([Voe96, Def.3.1.1]). A morphism of schemes $p : X \to Y$ is called a *topological epimorphism* if the underlying topological space of Y is a

[1] This fact is already known as it also follows from Rydh's more general refinement result concerning subtrusive covers. See [Ryd10, Thm. 3.12, Thm. 8.4].

quotient space of the underlying topological space of X. That is , p is surjective and a subset U of Y is open if and only if $p^{-1}(U)$ is open in X.

A topological epimorphism $p : X \to Y$ is called a *universal topological epimorphism* if for any morphism $f : Z \to Y$ the projection $Z \times_Y X \to Z$ is a topological epimorphism.

Remark 3.1.2. In the terminology of [Gro71] topological epimorphisms are (instead) called *submersive* and the universal topological epimorphisms are called *universally submersive.*

Example 3.1.3.

1. Using only point-set topology, we see that any open or closed surjective morphism is a topological epimorphism.

2. Since any flat morphism locally of finite presentation is universally open (See [Stacks, Tag 01UA]) and since surjectivity is stable under base change, we see that surjective flat morphisms locally of finite presentation are universal topological epimorphisms (This is also the case for surjective quasi-compact flat morphisms, see [Stacks, Tag 02JY]).

3. Proper morphisms are by definition universally closed, hence surjective proper morphisms are universal topological epimorphisms.

The following Lemma is straight forward.

Lemma 3.1.4. *Any composition of (universal) topological epimorphisms is a (universal) topological epimorphism.*

The h-topology

Definition 3.1.5 ([Voe96, Def. 3.1.2])**.** The h-pre-topology is the Grothendieck pre-topology on the category of schemes with coverings of the form $\{p_i : U_i \to X\}$ where $\{p_i\}$ is a finite family of morphisms of finite type such that the induced morphism $\coprod U_i \to X$ is a universal topological epimorphism. If we in addition require the p_i to be quasi-finite then we get the coverings of the qfh-pretopology.

The h-topology (resp. qfh-topology) is the Grothendieck topology associated to the h-pre-topology (respectively the qfh-pre-topology).

Example 3.1.6.

1. It follows immediately from the definitions that a coproduct of flat morphisms is flat, hence any flat covering (of finite presentation) is an h-covering. Moreover if a scheme S is quasi-compact and quasi-separated then it follows easily from Theorem 3.1.14 that a flat covering of S is a covering in the saturation of the qfh-pretopology (See Definition D.2.2).

2. Since open embeddings are flat, it follows from (1) that open finite coverings are h-coverings, hence on the category of Noetherian schemes the Zariski-topology is coarser than the h-topology.

3. Any surjective proper morphism is an h-covering.

4. Since the fibered product of a coproduct with a scheme over some other scheme, is the coproduct of the fibered products, we see that if $\{f_i : U_i \to X\}_{i\in I}$ is a jointly surjective finite family of proper morphisms, then this is an h-covering. In particular jointly surjective families of closed embeddings amd finite morphisms are h-coverings.

5. Consider the affine plane $\mathbb{A}^2_k$. Let $C = V(y)$ denote the x-axis and let U be the complement $\mathbb{A}^2_k \setminus C = D(y)$. The canonical morphism

$$\pi : U \coprod C \to \mathbb{A}^2_k$$

is clearly surjective, however $\pi^{-1}(C)$ is open while C is not open, hence π is not an h-covering.

6. By blowing up the affine plane at the origin and then removing a point from the exceptional divisor we get a morphism $p : U \to X$ which can be shown to be surjective but not a topological epimorphism. This counter example is a special case of Corollary 3.1.9 below.

Example 3.1.7. Consider the two k-algebra maps $\phi_1, \phi_2 : k[t] \to k[t]/(t^2)$ where the first is the canonical quotient and ϕ_2 is given by $t \mapsto 0$. When we compose either of these two maps with the canonical map $q : k[t]/(t^2) \to k$ we get the map $k[t] \to k$ given by $t \mapsto 0$. This shows that

$$\mathrm{Hom}_{\mathrm{Sch}\,/k}(\mathrm{Spec}(k[t]/(t^2)), \mathbb{A}^1_k) \to \mathrm{Hom}_{\mathrm{Sch}\,/k}(\mathrm{Spec}(k), \mathbb{A}^1_k)$$

is not injective as we have $\mathrm{Spec}(\phi_1) \circ \mathrm{Spec}(q) = \mathrm{Spec}(\phi_2) \circ \mathrm{Spec}(q)$. Note however that the closed embedding $\mathrm{Spec}(q) : \mathrm{Spec}(k) \to \mathrm{Spec}(k[t]/(t^2))$ is a qfh-covering, thus the presheaf $h_{\mathbb{A}^1_k}$ is not a sheaf in the qfh-topology. Which implies that both the qfh and the h topologies are not subcanonical.

Part (5) of 3.1.6 is a consequence of the following more general statement:

Proposition 3.1.8 ([Voe96, Proposition 3.1.3]). *Let $\{U_i \xrightarrow{p_i} X\}$ be an h-covering of a Noetherian scheme X. Denote by $\coprod_j V_j$ the disjoint union of irreducible components of $\coprod U_i$ such that for any j there exists an irreducible component X_i of X which is dominated by V_j. Then the morphism $q : \coprod V_j \to X$ is surjective.*

Proof. We recall the proof from loc.cit. Suppose first that X is irreducible. Let $x \in X$ be a point of X. We want to prove that x lies in the image of q. By considering the base change along the natural morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$, we may suppose that X is the spectrum of a local ring and x is the closed point of X.

Denote by Z the closure of the image Z' of those irreducible components of $\coprod U_i$ which are not dominant over X. Since Z' is a constructible subset of X by Chevalley's theorem ([Stacks, Tag 054J]), we have that $Z' = \coprod_k U'_k \cap C_k$ where U'_k are open and C_k are closed. Hence we have that

$$Z = \overline{Z'} = \bigcup_k \overline{U'_k \cap C_k} \subset \cup C_k$$

and since Z' does not contain the generic point, none of the C_k contain the generic point, thus

$$Z \subsetneq X.$$

From [GD67, (10.5.5) (i)] we have that the set of points x in X such that $\overline{\{x\}}$ is finite, is a dense subset X_0 of X. Note that since we are working in the spectrum of a local ring it follows that the closure of one-dimensional points consists of two points, hence the set of one dimensional points is contained in X_0. Now if x_0 is in X_0, then we have that $\overline{\{x_0\}}$ is finite and by [GD67, (10.5.3)] this implies that the dimension of $\overline{\{x_0\}}$ is less than or equal to one, hence the set X_0 is exactly the set of one dimensional points and the closed point, and so we have that the set of one-dimensional points of X is dense in X. Therefore there exists a one-dimensional point $y \in X$ which does not belong to Z. If x does not lie in the image of q then we have $q^{-1}(\overline{\{y\}}) = q^{-1}(\{y\} \cup \{x\}) = q^{-1}(\{y\})$ and so $q^{-1}(\{y\})$ is closed which implies that $p_i^{-1}(\{y\})$ is closed as well but $\{y\}$ is not closed in X, giving us a contradiction that $\{p_i\}$ is an h-covering.

Suppose now that X is an arbitrary scheme and let $X_{red} = \cup X_k$ be the decomposition of the maximal reduced subscheme of X into the union of its irreducible components. Consider the natural morphism $X_k \to X$ and let $\{U_i \times_X X_k \to X_k\}$ be the preimages of our h-covering. Then the morphisms $\coprod V_{j,k} \to X_k$, where $V_{j,k}$ are the irreducible components of $\coprod U_i \times_X X_k$ which are dominant over X_k are surjective, implying that $\coprod V_j \to X$ is surjective since $\coprod V_j = \coprod \coprod V_{j,k}$.

□

Corollary 3.1.9 ([Voe96, Rmk after 3.1.3])**.** *Let Z be a closed subscheme of an integral scheme X and $b : X_Z \to X$ the blow up with center Z. Suppose that for an open subscheme $U \hookrightarrow X_Z$, the composition $U \to X_Z \to X$ is an h-covering. Then $U = X_Z$.*

Proof. We expand on the explanation given in loc.cit. Consider the base change along the projection $X_Z \to X$, $X_Z \times_X X_Z$. Now since b is a birational morphism we have some open subset V of X such that $b^{-1}(V) \to V$ is an isomorphism and $b^{-1}(V) \cong b^{-1}(V) \times_X b^{-1}(V) \hookrightarrow X_Z \times_X X_Z$. Thus the closure of $b^{-1}(V) \times_X b^{-1}(V)$ in $X_Z \times_X X_Z$ denoted by T, is the unique irreducible component of $X_Z \times_X X_Z$ dominating X_Z. Now we claim that T is contained in the diagonal Δ. To see this, just note that the diagonal morphism $\Delta_{X_Z/X}$ restricted to $b^{-1}(V)$ yields the isomorphism $b^{-1}(V) \cong b^{-1}(V) \times_X b^{-1}(V)$, hence the diagonal is a closed subscheme containing T. Now from this it follows from Proposition 3.1.8 that $(U \times_X X_Z) \cap \Delta \to X_Z$ is a surjection, and so we must have $U = X_Z$. □

In the case where X is an irreducible geometrically unibranched scheme (this is the case whenever X is normal and connected) Proposition 3.1.8 has a converse.

Proposition 3.1.10 ([Voe96, Prop. 3.1.4]). *Let $\{p_i : U_i \to X\}$ be a finite family of quasi-finite morphisms over an irreducible geometrically unibranced Noetherian scheme X. Then $\{p_i\}$ is a qfh-covering if and only if the subfamily $\{q_j\}$ consisting of those p_i which are dominant over X is such that $\coprod q_j$ is surjective. In that case $\{q_j\}$ is also a qfh-covering of X.*

Proof. The "only if" part follows immediately from Proposition 3.1.8. The "if" part follows easily from Proposition 1.1.20 Item 3. □

Refinements of qfh-coverings

Let us recall the following theorem due to Grothendieck:

Theorem 3.1.11 (Zariski's main theorem). *Let $f : X \to S$ be a quasi-finite and separated morphism of schemes*[2] *and suppose that S is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd} \tag{3.1.1}$$

where j is a quasi-compact open embedding and π is finite.

Proof. See [GD67, Corollary (18.12.13)] or [Stacks, Tag 05K0]. □

As one might expect, Zariski's main theorem can be used to give neat refinements of qfh-coverings. In the case of Nagata schemes this is done in [SV96, Lemma 10.3]. Using the methods from Section 1.8 we can in fact remove the Nagata hypothesis.

[2]Recall that we are assuming all schemes to be separated anyway, so we could have omitted writing this assumption.

Proposition 3.1.12. *Let $Y \in \mathrm{Sch}/S$ be a Noetherian scheme and*

$$\{p_j : Y_j \to Y\}_{j\in J}$$

a qfh-covering. Then there exists a refinement of the form

$$\{Y_i' \to Y' \xrightarrow{p} Y\}_{i\in I} \tag{3.1.2}$$

where $\{Y_i' \to Y'\}_{i\in I}$ is a Zariski covering and $p : Y' \to Y$ is finite surjective.

Proof. If $\{Z_k\}_{k=1}^r$ denotes the irreducible components of Y then since the morphism $\coprod Z_k \to Y$ is finite surjective, it is enough to prove that the statement of the proposition for the covering obtained by base change to an arbitrary irreducible component of Y. In other words we can assume that Y is an integral Noetherian scheme.

Let $\{V_l\}_{l\in L}$ denote the set of irreducible components V_l of some Y_j such that V_l dominates Y. Since the morphisms $V_l \to Y_j \to Y$ are quasi-finite and separated it follows from Zariski's main theorem (Theorem 3.1.11) that there exists a factorization of the form

$$V_l \hookrightarrow \overline{V_l} \xrightarrow{\pi_l} Y \tag{3.1.3}$$

where the first map is an open embedding and π_l is finite surjective. Let E be the normal closure of the composite of all the extensions $k(\overline{V_l})/k(Y)$ and let $\nu : W \to Y$ denote the normalization of Y in the morphism $\mathrm{Spec}(E) \to Y$. Further let Y^n denote the normalization of Y and using Lemma 1.8.10 we write Y^n as

$$Y^n = \lim_{\theta} Y_\theta \tag{3.1.4}$$

with $Y_\theta \to Y$ finite and birational. Consider the covering of Y_θ obtained by base change

$$\{Y_j \underset{Y}{\times} Y_\theta \to Y_\theta\}_{j\in J} \tag{3.1.5}$$

and for any irreducible component V_{l_θ} of some $(Y_j \times_Y Y_\theta)$ dominating Y_θ apply Zariski's main theorem again to obtain a factorization of the form

$$V_{l_\theta} \hookrightarrow \overline{V_{l_\theta}} \xrightarrow{\pi_{l_\theta}} Y_\theta. \tag{3.1.6}$$

Now since the map $Y_\theta \to Y$ is birational it follows that if we let $\{V_{l_\theta}\}$ denote the set of irreducible components V_{l_θ} of some $Y_\theta \times_Y Y_j$ dominating Y_θ then the sets $\{V_{l_\theta}\}$ and $\{V_l\}$ have the same finite cardinality. Moreover for every such V_{l_θ} there is some V_l with the same function field. Thus applying [Stacks, Tag 035I] we get surjective integral Y-morphisms

$$\phi_{l_\theta} : W \to \overline{V_{l_\theta}},\ \nu_N : W \to Y^n,\ \nu_\theta : W \to Y_\theta \tag{3.1.7}$$

fitting into the following commutative diagram:

$$
\begin{tikzcd}
 & & \overline{V_{l_\theta}} \arrow[d, "\pi_{l_\theta}"] \\
W \arrow[rru, bend left, "\phi_{l_\theta}"] \arrow[r, "\nu_N"] \arrow[rr, bend right, "\nu_\theta"'] & Y^n \arrow[r] & Y_\theta .
\end{tikzcd}
\tag{3.1.8}
$$

Now for each l_θ (here we consider all θ) set

$$W_{l_\theta} := \phi_{l_\theta}^{-1}(V_{l_\theta}). \tag{3.1.9}$$

Letting $G = \mathrm{Aut}_{K(Y)}(E)$ it follows from the universal property of normalization in E that G-acts on Y_θ-automorphisms of W for every θ. For each $\sigma \in G$ we set

$$W_{l_\theta,\sigma} := \sigma(W_{l_\theta}). \tag{3.1.10}$$

Note that since G acts on Y_θ-automorphisms for every θ the following diagram is commutative

$$
\begin{tikzcd}
W_{l_\theta,\sigma} \arrow[rr, hook, "\circ"] \arrow[d, "id"] & & W \arrow[rr, "\nu_\theta"] & & Y_\theta \arrow[d, "id"] \\
W_{l_\theta,\sigma} \arrow[r, hook, "\circ"] & W \arrow[r, "\sigma^{-1}"] & W \arrow[r, "\phi_{l_\theta}"] & \overline{V_{l_\theta}} \arrow[r, "\pi_{l_\theta}"] & Y_\theta
\end{tikzcd}
\tag{3.1.11}
$$

Furthermore by construction the composition given in the bottom row factors through the following composition

$$V_{l_\theta} \overset{\circ}{\hookrightarrow} \overline{V_{l_\theta}} \xrightarrow{\pi_{l_\theta}} Y_\theta . \tag{3.1.12}$$

in particular the map $W_{l_\theta,\sigma} \to Y$ must necessarily factor through one of the p_j. Since the map $\nu : W \to Y$ is integral we can by Lemma 1.8.9 write it as a directed limit

$$W = \lim_{\lambda} W_\lambda \tag{3.1.13}$$

with $W_\lambda \to Y$ a finite morphism. Since π_{l_θ} is of finite presentation over Y it follows from Proposition 1.8.8 that for any $\sigma \in G$ there is some λ such that the map $\phi_{l_\theta} \circ \sigma^{-1} : W \to \overline{V_{l_\theta}}$ factors as the projection $W \to W_\lambda$ followed by a map $\phi_\lambda : W_\lambda \to \overline{V_{l_\theta}}$. Furthermore by Lemma 1.8.7 and using that the limit is directed we can also assume that there is some open subset U of W_λ, such that

$$W_{l_\theta,\sigma} = U \underset{W_\lambda}{\times} W. \tag{3.1.14}$$

Thus it follows that the composition $U \to W_\lambda \to Y$ factors through one of the p_j. Observe that since all the projections $W \to W_\lambda$ are surjective and the limit $W = \lim W_\lambda$ is directed we could conclude the proof if we knew that W could be covered by finitely many of the $W_{l_\theta,\sigma}$. In fact since W is quasi-compact (being given as an integral morphism to a Noetherian scheme) it is enough to prove the equality

$$W = \cup W_{l_\theta,\sigma}. \tag{3.1.15}$$

By [Bou64, Ch.5, Sec. 2, n.3, Prop. 6] we have that G acts transitively on the (set theoretic) fibers of ν_N. Hence in order to prove the claimed equality it is enough to show that every fiber of ν_N intersects one of the W_{l_θ}. To this extent pick an arbitrary $y' \in Y^n$. By Lemma 1.8.10 we have some θ and an $y'_\theta \in Y_\theta$ such that y' is the only point of Y^n lying over y'_θ. Furthermore by Proposition 3.1.8 the element y'_θ is in the image of at least one of the maps $\pi_{l_\theta}|_{V_{l_\theta}}$ thus if $v \in V_{l_\theta}$ lies over y'_θ then ν_N must necessarily map the fiber $\phi_{l_\theta}^{-1}(\{v\})$ to y' completing the proof. □

Remark 3.1.13. Proposition 3.1.12 is a special case of [Ryd10, Theorem 3.11] which is a generalization to the case of quasi-finite universally subtrusive morphisms. Our proof is much more similar to Voevodsky's original in the Nagata case.

Refining h-coverings

Theorem 3.1.9 of [Voe96] states that an h-covering of a Noetherian excellent[3] scheme has a refinement of the form

$$\{U_i \to U \to V \to X_{red} \to X\}_{i\in I} \tag{3.1.16}$$

where $\{U_i \to U\}_{i\in I}$ is an open covering of U and $U \to V$ is finite surjective, $V \to X_{red}$ is a blow-up of X_{red} in a closed subscheme and $X_{red} \to X$ is the obvious closed embedding. The proof reduces to the case of qfh-coverings which is where Voevodsky uses the excellency/Nagata assumption, however as we saw in Proposition 3.1.12 we do not need the Nagata hypothesis. In order to prove Voevodsky's theorem we need the following result which tells us that a faithfully flat morphism can be refined by a qfh-covering.

Theorem 3.1.14. *Suppose that $f : X \to S$ is a faithfully flat morphism locally of finite presentation. Then there exists a morphism $g : S' \to S$ which is faithfully flat, locally of finite presentation and locally quasi-finite and an S-morphism $S' \to X$ factorizing g together with f. If S is quasi-compact (resp. quasi-compact and quasi-separated which in particular is the case if S is Noetherian), then S' can be taken to be an affine scheme (resp. S' can be taken to be affine and g can be taken to be quasi-finite).*

Proof. See [GD67, Corollaire 17.16.2]. □

[3] The reader not familiar with this notion need not worry, all that is needed is that such schemes are Nagata, and we will actually see that this assumption is not necessary.

Remark 3.1.15 (Fun implication)**.** Hilbert's Nullstellensatz may be stated as follows: For any finite type morphism $X \to \operatorname{Spec}(k)$ where X is a scheme over a field k , there exists a finite field extension k'/k and a morphism of k-schemes $\operatorname{Spec} k' \to X$.

As quasi-finite morphisms over a field are finite we get the Nullstellensatz as a special case of Theorem 3.1.14.[4]

Theorem 3.1.16. *Let Y be a Noetherian scheme. Then any h-covering $\{Y_j \to Y\}_{j\in J}$ has a refinement of the form*

$$\{Y_i' \to Y' \xrightarrow{p} Y\}_{i\in I} \tag{3.1.17}$$

where $\{Y_i' \to Y'\}_{i\in I}$ is a Zariski covering and $p : Y' \to Y$ is a proper cover. Furthermore the map $p : Y' \to Y$ factors as a finite surjective morphism $Y' \to Y''$ followed by a blow-up $Y'' \to Y_{red}$ followed by the closed embedding $Y_{red} \to Y$.

Proof. Proposition 3.1.12 makes it possible to run the original proof due to Voevodsky which we now recall:

Suppose that $\{p_j : Y_j \to Y\}_{j\in J}$ is an h-covering of Y. Since the evident map $Y_{red} \to Y$ is an h-covering base change allows us to reduce to the case where Y is reduced. By [Stacks, Tag 052B] there is a dense open subscheme Y_0 of Y such that the morphism $\coprod_j p_j : \coprod Y_j \to Y$ is flat over Y_0 (hence p_j is also flat over Y_0). Now from Theorem 1.2.3 we can find a closed subscheme Z disjoint with Y_0 such that if Y_Z denotes the blow-up with center Z then the strict transform $(\tilde{\coprod Y_j})$ is flat over Y_Z or in other words the composition

$$f : (\tilde{\coprod Y_j}) \hookrightarrow Y_Z \times_Y (\coprod Y_j) \longrightarrow Y_Z \tag{3.1.18}$$

is flat. Let C denote the scheme theoretic closure of the complement of the strict transform $(\tilde{\coprod Y_j})$ in $Y_Z \times_Y \coprod Y_j$ and note that the following set of maps

$$\{f,\ C \to Y_Z\} \tag{3.1.19}$$

is an h-covering of Y_Z. Note that since the closed embedding

$$(\tilde{\coprod Y_j}) \hookrightarrow Y_Z \times_Y (\coprod Y_j) \tag{3.1.20}$$

is an isomorphism over the generic points of Y_Z it follows then from Chevalley's Theorem ([Stacks, Tag 054J]) and [Stacks, Tag 005K] that C cannot dominate any irreducible component of Y_Z thus by Proposition 3.1.8 it follows that f is faithfully flat (hence an h-covering by Example 3.1.6 Item 1). We can now apply Theorem 3.1.14 to obtain a faithfully-flat quasi-finite morphism $U \to Y_Z$

[4] This example is also given in [Mil80].

which factors through f. By considering the preimages of Y_j in U which we denote by U_j we thus obtain a refinement of the original covering of the form

$$\{U_j \to U \to Y_Z \to Y\}_{j \in J} \tag{3.1.21}$$

such that

$$\{U_j \to U \to Y_Z\}_{j \in J} \tag{3.1.22}$$

is a qfh-covering of the Noetherian scheme Y_Z. Proposition 3.1.12 now allows us to conclude the proof. □

Remark 3.1.17. Theorem 3.1.16 can also be deduced from the more general result [Ryd10, Thm. 3.12] (see also [Ryd10, Thm. 8.4]).

Applications of the refinement result

A first immediate corollary of Theorem 3.1.16 is re-obtaining a Theorem originally proved by Goodwillie in [GL01];

Corollary 3.1.18 (Goodwillie). *Let X be a Noetherian scheme. Then every universal topological epimorphism $U \to X$ of finite type is a ph-cover*[5]

We also obtain another proof of the following useful known result:

Lemma 3.1.19 ([Gro71, Exp. IX, Prop.2.4], [SV00, Lem. 4.1.3]). *Let S be a Noetherian scheme and $p : X \to S$ a scheme of finite type over S. Suppose that there is an h-covering $f : S' \to S$ such that the scheme $X' = X \times_S S'$ is proper over S'. Then X is proper over S.*

Proof. If $S' \to S$ is an h-covering such that $X' \to S'$ is proper then since the property of being proper is local on the target it follows easily from Theorem 3.1.16 that there is an h-covering $\overline{X} \to X$ such that the composition $\overline{X} \to X \to S$ is proper. From this it follows that $X \to S$ must be proper. □

Another nice application of the refinement result is the following characterisation of universal topological epimorphisms of finite type which can also be deduced by other means:

Proposition 3.1.20 ([Gro71, Exp. IX, Rem. 2.6], [Ryd10, Cor. 2.10]). *Let S be a Noetherian scheme and $f : X \to S$ a morphism of finite type. Then the following statements are equivalent.*

1. *The morphism f is an h-covering.*

[5] The ph topology is the topology generated by Zariski covers and proper surjections.

2. *For any discrete valuation ring R and diagram of solid arrows*

$$\begin{tikzcd} \mathrm{Spec}(R') \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & S \end{tikzcd} \tag{3.1.23}$$

there is a discrete valuation ring R' and morphisms making the diagram commutative and such that the left vertical morphism is surjective.

Proof. If f is an h-covering then it follows easily from Theorem 3.1.16 and Lemma 1.3.12 that for any morphism $\mathrm{Spec}(R) \to S$ with R a discrete valuation ring the diagram given in (3.1.23) can be filled with R' a discrete valuation ring and $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ being surjective.

To prove the converse statement note that it is enough to prove the weaker statement that if (2) holds then f is necessarily a topological epimorphism since property (2) is obviously preserved by base change. To this extent suppose for the sake of contradiction that (2) holds but f is not a topological epimorphism. Then we have a non-open subset U of S such that $f^{-1}(U)$ is open. By assumption the morphism f is obviously surjective and it follows from [Stacks, Tag 054J] that $U = f(f^{-1}(U))$ is a constructible subset which is not open hence it cannot be stable under generalization ([Stacks, Tag 0542]). Thus there is a point $s' \in U$ and a point $s \in S \setminus U$ such that s' is contained in the closure of the point s. By Corollary 2.1.3 we can find a discrete valuation ring R and a map $\mathrm{Spec}(R) \to S$ such that the generic point of $\mathrm{Spec}(R)$ maps to s and the closed point maps to s'. By assumption there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(R') \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & S \end{tikzcd}$$

where the image x of the generic point of $\mathrm{Spec}(R')$ in X is a generalization of the image of the closed point which by assumption is contained in $f^{-1}(U)$. Since this set is open it follows that we must have $x \in f^{-1}(U)$ hence $s \in U$ giving the desired contradiction. □

Remark 3.1.21. One can in fact avoid Theorem 3.1.16 and work with the definition of the h-topology to deduce both Lemma 3.1.19 and Proposition 3.1.20. However we find the proofs slightly easier using Theorem 3.1.16.

Proposition 3.1.20 gives further means of checking flatness of morphisms:

Proposition 3.1.22 (h-topological criterion for flatness)**.** *Let S be a reduced Noetherian scheme and $f : X \to S$ be a morphism locally of finite type. If there exists an h-covering $S' \to S$ such that the induced morphism $f' : S' \times_S X \to S'$ is flat then f is flat.*

Proof. Suppose that $S' \to S$ is an h-covering making the induced map $f' : S' \times_S X \to S'$ flat. To show that f is flat it is by Theorem 1.3.14 enough to show that if $\operatorname{Spec}(R) \to S$ is any morphism from a discrete valuation ring then the induced map $X_R = X \times_S \operatorname{Spec}(R) \to \operatorname{Spec}(R)$ is flat. Given such a map $\operatorname{Spec}(R) \to S$ we apply Proposition 3.1.20 to obtain a diagram

$$\begin{tikzcd} \operatorname{Spec}(R') \arrow[r] \arrow[d, "g"] & S' \arrow[d] \\ \operatorname{Spec}(R) \arrow[r] & S \end{tikzcd} \tag{3.1.24}$$

with R' a discrete valuation ring and g surjective. By Lemma 1.3.8 the map g is faithfully flat and by assumption $X_{R'} = X \times_S \operatorname{Spec}(R') \to \operatorname{Spec}(R')$ is flat hence we can apply [Stacks, Tag 036K] to conclude that $X_R = X \times_S \operatorname{Spec}(R) \to \operatorname{Spec}(R)$ is also flat. □

Remark 3.1.23. Proposition 3.1.22 is essentially a special case of [Pic86, Prop II.21, page 584].

Cousins of the h-topology

Throughout the literature there are many coarser variants of the h-topology which provide more control over the field arithmetic. To name a few one has the *cdh*-topology introduced in [SV00], the *rh* topology from [GL01] and the *eh*-topology found in [Gei06]. These topologies are slightly more awkward to define, and in fact for the purposes of this thesis it will be enough to only describe the finest sub-topology of the h-topology where points have liftings inducing separable field extensions. We will now begin to describe this topology in a more precise manner.

Definition 3.1.24. A morphism of schemes $f : X' \to X$ satisfies the *separable lifting condition* if for every point $x \in X$ there is a point $x' \in f^{-1}(x)$ such that the induced map of residue fields $k(x) \to k(x')$ is a separable field extension.

Lemma 3.1.25. *The separable lifting condition is preserved under composition and stable under base change.*

Proof. The first statement follows from Lemma 1.4.13 and the second from Lemma 1.4.14. □

Definition 3.1.26. The *sd-h* pre-topology on the category of schemes is the pre-topology where coverings $\{p_i : X_i \to X\}_{i \in I}$ are h-coverings such that the induced morphism

$$\coprod_{i \in I} X_i \to X \tag{3.1.25}$$

satisfies the separable lifting condition.

Remark 3.1.27. It can be shown that if the separable lifting condition holds for a morphism $X' \to X$ of finite type, then the point x' over x can in fact be chosen such that the induced extension of residue fields is finite separable. Furthermore we expect our sd-h topology to coincide with the *sdh*-topology introduced in [HKK17]. The *sdh* topology is the coarsest saturated topology containing étale covers and proper surjective morphisms $p : X' \to X$ such that for every $x \in X$ there is a point $x' \in X'$ such that the induced map of residue fields $k(x) \to k(x')$ is a finite separable extension.

For the rest of this Chapter we restrict the h, qfh and sd-h topologies to the category of Noetherian schemes over a fixed Noetherian scheme S. This means that we only consider those coverings which are sets of morphisms of Noetherian S-schemes. All results proved so far (such as for instance Theorem 3.1.16) are still valid in this setup.

3.2 Chow sheaves in the h-topologies

Many of the Chow presheaves defined in 2.3 are actually sheaves in one of the h-topologies (h,qfh or sd-h). We start by showing that $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ is a separated presheaf with respect to the h-topology.

Lemma 3.2.1. *The presheaves* $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ *are separated with respect to the h-topology.*

Proof. Let $\{p_i : U_i \to S\}_{i \in I}$ be any h-covering of the scheme S and let k be a field and $x : \mathrm{Spec}(k) \to S$ be a k-point and (x_0, x_1, R) be any fat point over k. We can pick an $i \in I$ such that the (set theoretic) image of x_1 is contained in the (set theoretic) image of p_i. Using Corollary 2.1.3 we can find a field L an L-point $y : \mathrm{Spec}(L) \to U_i$ and a fat point (y_0, y_1, A) over y. Letting E be a composite of L and k we get a commutative diagram of the form

$$\begin{tikzcd}
 & \mathrm{Spec}(L) \arrow[r, "y_0"] & \mathrm{Spec}(A) \arrow[r, "y_1"] & U_i \arrow[d, "p_i"] \\
\mathrm{Spec}(E) \arrow[ur] \arrow[dr] & & & \\
 & \mathrm{Spec}(k) \arrow[r, "x_0"] & \mathrm{Spec}(R) \arrow[r, "x_1"] & S
\end{tikzcd}$$

which implies that if $\mathrm{cycl}(p_i)(\mathcal{Z}) = 0$ for all i then $(x_0, x_1)^*(\mathcal{Z}) = 0$ for any fat point and thus by Corollary 2.1.17 we have $\mathcal{Z} = 0$. ☐

Proving that $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ and $\mathrm{Cycl}^{eff}(X/S, r)_{\mathbb{Q}_+}$ are sheaves in the h-topology reduces to the following lemma which is stated but not proved in [SV00].

Lemma 3.2.2. *Let $p_Y : Y \to \mathrm{Spec}(k), p_X : X \to \mathrm{Spec}(k)$ be two schemes of finite type over a field k. Then the sequence of abelian groups*

$$\mathrm{Cycl}(X/\mathrm{Spec}(k), r) \otimes \mathbb{Q} \overset{\mathrm{cycl}(p_Y)}{\to} \mathrm{Cycl}(X \underset{\mathrm{Spec}(k)}{\times} Y/Y, r) \otimes \mathbb{Q} \overset{\mathrm{cycl}(pr_1)-\mathrm{cycl}(pr_2)}{\to}$$
$$\to \mathrm{Cycl}((X \underset{\mathrm{Spec}(k)}{\times} Y \underset{\mathrm{Spec}(k)}{\times} Y)/(Y \underset{\mathrm{Spec}(k)}{\times} Y), r) \otimes \mathbb{Q}$$

where $pr_i : Y \times_{\mathrm{Spec}(k)} Y \to Y$ are the projections, is exact.

Proof. Suppose that $\mathcal{Y}$ is in the kernel of the map $(\mathrm{cycl}(pr_1) - \mathrm{cycl}(pr_2))$. Since $Y \to \mathrm{Spec}(k)$ is of finite type we can find a normal extension L/k and a morphism of finite type k-schemes $t : \mathrm{Spec}(L) \to Y$. Let σ be any k-automorphism of L. We have a map $q_{t,\sigma} : \mathrm{Spec}(L) \to Y \times_{\mathrm{Spec}(k)} Y$ satisfying

$$t = pr_1 \circ q_{t,\sigma} \text{ and } t \circ \mathrm{Spec}(\sigma) = pr_2 \circ q_{t,\sigma}.$$

From our assumption on $\mathcal{Y}$ and Lemma 2.3.20 it then follows that

$$\mathrm{cycl}(t)(\mathcal{Y}) = \mathrm{cycl}(\mathrm{Spec}(\sigma))(\mathrm{cycl}(t)(\mathcal{Y})) = \sigma^*(\mathcal{Y})$$

where σ^* denotes the action on $\mathrm{Cycl}(X_L)$. Thus we have that the cycle $\mathrm{cycl}(t)(\mathcal{Y})$ is $\mathrm{Gal}(L/k)$-invariant and thus by Lemma 1.7.3 there is a unique cycle $\mathcal{Z} \in \mathrm{Cycl}(X/\mathrm{Spec}(k), r) \otimes \mathbb{Q}$ such that $\mathcal{Z} \otimes_k L = \mathrm{cycl}(t)(\mathcal{Y})$. Let now $y : \mathrm{Spec}(E) \to Y$ be an E-point of Y and (y_0, y_1, R) be a fat point over y. Since $\mathrm{Spec}(E) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ is non-empty we can find a field M and a morphism $\mathrm{Spec}(M) \to \mathrm{Spec}(E) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ and hence an induced morphism $\mathrm{Spec}(M) \to Y \times_{\mathrm{Spec}(k)} Y$ and by functoriality and the defining property of $\mathcal{Z}$ we have that the two cycles $\mathcal{Y}$ and $\mathrm{cycl}(p_Y)(\mathcal{Z})$ pull back to the same element in $\mathrm{Cycl}(X_M/\mathrm{Spec}(M), r)$. By Lemma 1.7.2(2) it then follows that we must have

$$(y_0, y_1)^*(\mathcal{Y}) = \mathrm{cycl}(y)(\mathcal{Y}) = \mathrm{cycl}(y)(\mathrm{cycl}(p_Y)(\mathcal{Z})) = (y_0, y_1)^*(\mathrm{cycl}(p_Y)(\mathcal{Z}))$$

and thus $\mathcal{Y}$ satisfies the defining property of $\mathrm{cycl}(p_Y)(\mathcal{Z})$ and we are done. □

Theorem 3.2.3. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ are sheaves in the h-topology.*

Proof. Apart from Lemma 3.2.1 and Lemma 3.2.2 all the details are given in [SV00, Theorem 4.2.2]. □

Corollary 3.2.4. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves $\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}$ are sheaves in the h-topology.*

Proof. Since $\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}$ is a subpresheaf of $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ it is enough to show that if $\{p_i : T_i \to T\}_{i \in I}$ is an h-covering such that if $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)_{\mathbb{Q}}(T)$ satisfies

$$\mathrm{cycl}(p_i)(\mathcal{Z}) \in \mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}(T_i) \tag{3.2.1}$$

for all i then we must have $\mathcal{Z} \in \mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}(T)$, which follows easily from Lemma 2.3.13 and Lemma 1.7.2. $\square$

The following Proposition is [SV00, Prop.4.2.6]. Theorem 3.1.16 allows us to simplify the original proof somewhat.

Proposition 3.2.5. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves* $\mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}$ *are sheaves in the h-topology.*

Proof. In view of Theorem 3.2.3 it is enough to show that if $\{p_i : T_i \to T\}_{i \in I}$ is an h-covering of the Noetherian scheme T and $\mathcal{Z} \in \mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ satisfies $\mathrm{cycl}(p_i)(\mathcal{Z}) \in \mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}(T_i)$ for all i then $\mathrm{Supp}(\mathcal{Z}) \subset T \times_S X$ is proper over T. By Theorem 3.1.16 the covering $\{p_i\}_{i \in I}$ has a refinement of the form

$$\{U_j \to T' \to T\}_{j \in J}$$

where $\{U_j \to T'\}_{j \in J}$ is an open covering and $T' \to T$ is proper surjective. Since properness is local on the target it is clear that $\mathrm{Supp}(\mathcal{Z}_{T'})$ is proper over T' and hence also over T. Furthermore Lemma 2.3.18 Item (2) tells us that $\mathrm{Supp}(\mathcal{Z}_{T'}) \to \mathrm{Supp}(\mathcal{Z})$ is surjective from which it easily follows that $\mathrm{Supp}(\mathcal{Z}) \to T$ must be proper. $\square$

Corollary 3.2.6 ([SV00, prop.4.2.6])**.** *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves* $\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}$ *are sheaves in the h-topology.*

Proof. In the category of presheaves we have the following pullback diagram

$$\begin{tikzcd}
\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+} \arrow[r, hook] \arrow[d, hook] & \mathrm{PropCycl}(X/S, r)_{\mathbb{Q}} \arrow[d, hook] \\
\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+} \arrow[r, hook] & \mathrm{Cycl}(X/S, r)_{\mathbb{Q}}.
\end{tikzcd} \tag{3.2.2}$$

Since the presheaves $\mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}$, $\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}$ and $\mathrm{Cycl}(X/S, r)_{\mathbb{Q}}$ are sheaves in the h topology by Proposition 3.2.5, Corollary 3.2.4 and Theorem 3.2.3 respectively, and the forgetful functor from sheaves to presheaves has a left adjoint hence commutes with limits it follows immediately that $\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+}$ must necessarily be a sheaf in the h-topology. $\square$

Definition 3.2.7. Let S be a scheme. We denote by $\mathrm{Exp.\,Char}(S)$ the set

$$\mathrm{Exp.\,Char}(S) := \{p \in \mathbb{N} : \ \exists s \in S \text{ with } \mathrm{exp.\,char}(k(s)) = p\},$$

where $\mathrm{exp.\,char}(k(s))$ denotes the exponential characteristic of the field $k(s)$.

Corollary 3.2.8. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then for any subring Λ of $\mathbb{Q}$ (resp. subsemi-ring of $\mathbb{Q}_+$) such that every element of* $\mathrm{Exp.Char}(S)$ *is invertible in Λ the presheaves*

$$\mathrm{Cycl}(X/S, r)_{UI} \otimes_{\mathbb{Z}} \Lambda, \ \mathrm{PropCycl}(X/S, r)_{UI} \otimes_{\mathbb{Z}} \Lambda$$

$$(\textit{resp.}\ \ \mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{UI} \otimes_{\mathbb{N}} \Lambda, \ \mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{UI} \otimes_{\mathbb{N}} \Lambda\ )$$

are sheaves in the h-topology.

Proof. Keeping in mind Item 2 of Proposition 1.7.7, the result is deduced from a technique which we are now familiar with. □

Theorem 4.2.9 of [SV00] says that the presheaves of relative cycles with universally integral coefficients are sheaves in the *cdh* topology. We will now show that they are in fact sheaves in the finer sd-h topology.

Theorem 3.2.9. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then the presheaves* $\mathrm{Cycl}(X/S, r)_{UI}$, $\mathrm{Cycl}^{\mathit{eff}}(X/S, r)_{UI}$, $\mathrm{PropCycl}(X/S, r)_{UI}$ *and* $\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{UI}$ *are sheaves in the sd-h topology.*

Proof. We only prove the case of $\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{UI}$; the other cases are proved mutatis mutandis. By Corollary 3.2.6 it is enough to show that if $\mathcal{Z} \in \mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}(T)$ and $\{p_i : T_i \to T\}_{i \in I}$ is an sd-h covering such that $\mathrm{cycl}(p_i)(\mathcal{Z}) \in \mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{UI}(T_i)$ for all i then we must have $\mathcal{Z} \in \mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{UI}(T)$. This follows immediately from Lemma 2.3.21 and the definition of the sd-h topology. □

For more fascinating results concerning Chow sheaves in the h-topologies, such as the fact that $\mathrm{Cycl}_{\mathrm{equi}}(X/S, r)_{\mathbb{Q}}$ are sheaves in the qfh topology or that the h-sheafification of the presheaf $\mathrm{PropCycl}^{\mathit{eff}}(X/S, r)_{\mathbb{Q}_+} \otimes_{\mathbb{N}} \mathbb{Q}$ gives $\mathrm{PropCycl}(X/S, r)_{\mathbb{Q}}$, see section 4.2 of [SV00].

3.3 Sheaves in the qfh-topology via finite group actions

In this section we recall that sheaves in the qfh-topology interact neatly with quotients by finite groups, an aspect of qfh-sheaves playing an important role in [SV96]. Despite this being the natural chapter to discuss such topics, the proof of Proposition 7.1.2 is actually the only part of the thesis where this technology will be put to use. Thus the reader may chose to skip this section on a first reading.

Finite group actions and fibers

The following useful lemma is stated but not proved in [SV96].

Lemma 3.3.1 ([SV96, Lemma 5.1])**.** *Let $q : X \to S$ be a finite surjective morphism of schemes with S Noetherian and let G be a finite group acting on the right on X/S. The following conditions are equivalent*

(1) *For any $s \in S$ the (induced) action of G on $q^{-1}(s)$ is transitive, for any $x \in q^{-1}(s)$ the field extension $k(x)/k(s)$ is normal and the natural homomorphism $G_x = \mathrm{Stab}_G(x) \to \mathrm{Gal}(k(x)/k(s))$ is surjective.*

(2) *For any algebraically closed field Ω and any geometric point $\zeta : \operatorname{Spec}\Omega \to S$ the action of G on the set of geometric points of X over ζ is transitive.*

(3) *For any algebraically closed field Ω and any geometric point $\zeta : \operatorname{Spec}\Omega \to S$ the action of G on the (underlying set of the) geometric fiber $X_\zeta = X \times_S \operatorname{Spec}\Omega$ is transitive.*

Proof. **(1) implies (2):** Suppose we have two geometric points $\tau_1, \tau_2 : \operatorname{Spec}\Omega \to X$ over $\zeta : \operatorname{Spec}\Omega \to S$. In other words we have a commutative diagram

$$\begin{tikzcd} & X \arrow[d, "q"] \\ \operatorname{Spec}\Omega \arrow[ur, shift left, "\tau_1"] \arrow[ur, shift right, "\tau_2"'] \arrow[r, "\zeta"'] & S \end{tikzcd}$$

We want to show that there is some $g \in G$ such that $\tau_2 = g \circ \tau_1$. Let ω denote the only point of Ω. Since G acts transitively on $q^{-1}(\zeta(\omega))$ we may assume that $\tau_1(\omega) = \tau_2(\omega)$. The problem now translates to having a commutative diagram of fields

$$\begin{tikzcd} & k(x) \arrow[dl, shift right, "\sigma_1"'] \arrow[dl, shift left, "\sigma_2"] \\ \Omega & k(s) \arrow[l] \arrow[u] \end{tikzcd}$$

where $k(x)/k(s)$ is normal by assumption, and we want to find $g \in G_x \subset G$ such that the canonically induced morphism $\overline{g} : k(x) \to k(x)$ has the property that

$$\sigma_2 = \sigma_1 \circ \overline{g}.$$

By [Stacks, Tag 0BR4] we have some $h \in \mathrm{Gal}(k(x)/k(s))$ such that

$$\sigma_2 = \sigma_1 \circ h$$

and since the canonical map $G_x \to \mathrm{Gal}(k(x)/k(s))$ is surjective by assumption, we have some $g \in G_x$ such that $\overline{g} = h$.

For (2) implies (3): The action of G on the geometric fiber is given as follows. Let $p : X_\Omega \to X$ be the projection to X. Then for any $g \in G$, by the universal property of X_Ω, we have a unique map $\rho_g : X_\Omega \to X_\Omega$ such that

$$p \circ \rho_g = g \circ p$$

and ρ_g is an automorphism with inverse $\rho_{g^{-1}}$. It is clear that this gives a right group action of G on the geometric fiber X_Ω. Now this again gives a right group action of the underlying set of X_Ω by $(x, g) \mapsto \rho_g(x) \in X_\Omega$. We now want to show that if $x_1, x_2 \in X_\Omega$, then we can find some $g \in G$ such that $\rho_g(x_1) = x_2$. Since $X_\Omega \to \operatorname{Spec}\Omega$ is finite and Ω is algebraically closed, we have that giving a point in X_Ω is the same as giving a Ω-morphism $\operatorname{Spec}\Omega \to X_\Omega$. We denote the morphisms corresponding to x_1 resp. x_2, be $\overline{x_1} : \operatorname{Spec}\Omega \to X_\Omega$ resp. $\overline{x_2}$.

By the assumption of (2) there is some $g \in G$ such that

$$g \circ (p \circ \overline{x_1}) = p \circ \overline{x_2}.$$

But then

$$p \circ (\rho_g \circ \overline{x_1}) = p \circ \overline{x_2}$$

thus by universal property of X_Ω we must have

$$\rho_g \circ \overline{x_1} = \overline{x_2}.$$

Thus $\rho_g(x_1) = x_2$.

For (3) implies (2): Suppose that η is the point of $\operatorname{Spec}\Omega$ and let $s = \zeta(\eta) \in S$. We want to show that if $\tau_1, \tau_2 : \operatorname{Spec}\Omega \to X$ are two geometric points over ζ, then we can find some $g \in G$ such that $g \circ \tau_1 = \tau_2$. By properties of the fiber product, we may replace X with the finite scheme $q^{-1}(s)$ and S with $\operatorname{Spec} k(s)$. Let $p : X_\Omega \to q^{-1}(s)$ be the projection. By universal property we get induced maps $\tau_1', \tau_2' : \operatorname{Spec}\Omega \to X_\Omega$ with $p \circ \tau_i' = \tau_i$ for $i = 1, 2$. By assumption there is some $g \in G$ such that $\rho_g \circ \tau_1' = \tau_2'$. Further we have that

$$\tau_2 = p \circ \tau_2' = p \circ (\rho_g \circ \tau_1') = g \circ p \circ \tau_1' = g \circ \tau_1$$

which completes the proof of the implication.

For (3) implies (1): Assume that (1) fails and show that (3) fails by splitting it up into the three cases where at least one of the assumptions of (1) fails. This is not hard given what we have already done, thus we omit the proof of this last implication. □

Notation 3.3.2. From now on if we say that $(X, G)/S$ or $(X \to S, G)$ satisfies the equivalent conditions of Lemma 3.3.1 then we mean that X is a scheme finite over S, G a finite group acting on S-automorphisms and X/S together with the action of G satisfy the equivalent conditions of Lemma 3.3.1.

Corollary 3.3.3 ([SV96, Corollary 5.2]). *Assume that the equivalent conditions of Lemma 3.3.1 are fulfilled and assume further that S is irreducible. Then G acts transitively on the set of irreducible components of X and each component maps surjectively onto S.*

Proof. First note that since q is closed and surjective and S is irreducible we have that some irreducible component X_1 of X is mapped onto S. Now suppose that X_2 is an irreducible component of X such that that $q(X_2) \neq X$. Let η_2 denote the generic point of X_2 and let ξ be the generic point of S. We have that any affine open subset of S containing $q(\eta_2)$ must also contain ξ, and from the going up property of integral ring morphisms, we deduce that there is a point $x \in X_1$ with

$$x \in q^{-1}(q(\eta_2)).$$

Thus x is not a generic point of X, but since (1) in Lemma 3.3.1 is fulfilled, we have some $g \in G$ such that $g(x) = \eta_2$, but then since $\overline{\{x\}}$ is not an irreducible component of X, X_2 cannot be an irreducible component either which is a contradiction. Thus we have deduced that all the generic points of X are contained in the fiber $q^{-1}(\xi)$, and since G acts transitively on this fiber we are done. $\square$

Corollary 3.3.4. *Assume that $(X, G)/S$ satisfies the equivalent conditions of Lemma 3.3.1. Then the graph of the pair $(X, G)/S$, $\psi_{X/S} : G_X \to X \times_S X$ (Definition 1.5.3) is finite and surjective.*

Proof. It is enough to show that $\psi_{X/S}$ is surjective over each $s \in S$. To this extent pick some $s \in S$ and let Ω be an algebraic closure of the field $k(s)$. Consider the commutative diagram

$$\begin{tikzcd}
G_{X_\Omega} \arrow[r, "\psi_{X_\Omega/\operatorname{Spec}\Omega}"] \arrow[d] & X_\Omega \times_{\operatorname{Spec}\Omega} X_\Omega \arrow[d] & \\
G_{X_s} \arrow[r, "\psi_{X_s/\operatorname{Spec}k(s)}"] \arrow[d] & X_s \times_{\operatorname{Spec}k(s)} X_s \arrow[r, "\cong"] \arrow[d] & (X \times_S X)_s \arrow[dl] \\
G_X \arrow[r, "\psi_{X/S}"] & X \times_S X &
\end{tikzcd}$$

Since $(X_\Omega \times_{\operatorname{Spec}\Omega} X_\Omega) \to X_s \times_{\operatorname{Spec}k(s)} X_s$ factors as

$$X_\Omega \underset{\operatorname{Spec}\Omega}{\times} X_\Omega \overset{\cong}{\to} (X_s \underset{\operatorname{Spec}k(s)}{\times} X_s)_{\operatorname{Spec}\Omega} \to X_s \underset{\operatorname{Spec}k(s)}{\times} X_s$$

where the last morphism is surjective, we have that $(X_\Omega \times_{\operatorname{Spec}\Omega} X_\Omega) \to X_s \times_{\operatorname{Spec}k(s)} X_s$ is surjective, hence it is enough to prove surjectivity of $\psi_{X_\Omega/\operatorname{Spec}\Omega}$. To this extent note that giving a point in $X_\Omega \times_{\operatorname{Spec}\Omega} X_\Omega$ is equivalent to giving a $\operatorname{Spec}\Omega$-point $\operatorname{Spec}\Omega \to X_\Omega \times_{\operatorname{Spec}\Omega} X_\Omega$ which again is equivalent to giving two $\operatorname{Spec}\Omega$-points of X_Ω, say x, y. Now let $\sigma \in G$ be such that $x \cdot \sigma = y$. Then the composition

$$\operatorname{Spec}\Omega \xrightarrow{x} (X_\Omega)_\sigma \to G_{X_\Omega} \overset{\psi_{X_\Omega/\operatorname{Spec}\Omega}}{\to} X_\Omega \underset{\operatorname{Spec}\Omega}{\times} X_\Omega$$

is the point corresponding to (x, y). ☐

Lemma 3.3.5. *Suppose that $(X,G)/S$ satisfies the equivalent conditions of Lemma 3.3.1 . If $Y \to S$ is any morphism then $Y \times_S X \to Y$ together with the canonically induced action of G also satisfy the equivalent conditions of Lemma 3.3.1.*

Proof. Let $\operatorname{Spec}\Omega \to Y$ be a geometric point of Y. Then $\operatorname{Spec}(\Omega) \times_Y (Y \times_S X) \cong \operatorname{Spec}\Omega \times_S X$ which G acts transitively on by assumption. ☐

Corollary 3.3.6. *Suppose $(X \to S, G)$ satisfies the equivalent conditions of Lemma 3.3.1 and suppose we have a morphism $q : Y \to X$. For each $\sigma \in G$ the morphisms $id_Y : Y \to Y$ and $\sigma \circ q : Y \to X$ induce a morphism $Y \to Y \times_S X$ and in turn a morphism*

$$\upsilon_{Y,X/S} : \coprod_{\sigma \in G} Y \to Y \underset{S}{\times} X.$$

The aforementioned morphism $\upsilon_{Y,X/S}$ is finite and surjective.

Proof. The morphism $\upsilon_{Y,X/S}$ is clearly finite. Hence it is enough to prove surjectivity. To this extent let $y \in Y$ be any point in Y and let $t \in Y \times_S X$ be any point lying over y. Letting y' denote the image of y in $Y \times_S X$ under the morphism $Y \to Y \times_S X$ induced by $id_Y : Y \to Y$ and q , we have that y' is in the fiber of y. Now by Lemma 3.3.5 we have that the induced action of G on $Y \times_S X$ acts transitively on fibers. Hence there is some $\sigma \in G$ such that

$$(id_Y \times \sigma)(y') = t,$$

but this is also the image of y under the morphism $Y \to Y \times_S X$ induced by id_Y and $\sigma \circ q$. ☐

Example 3.3.7. Assume that S is a normal connected (hence integral) scheme. Let E be a finite normal extension of the field $K(S)$ with Galois group $G = \operatorname{Gal}(E/K(S))$, let Y denote the normalization of S in E and let $q : Y \to S$ be the normalization morphism.

From [Bou64, Ch.5, Sec. 2, n.3, Prop. 6] it follows that (q, G) satisfies the equivalent conditions of Lemma 3.3.1.

Example 3.3.8. Suppose that X is a scheme of finite type over Z and suppose that the finite group G acts admissibly on X by Z-automorphisms. Then by Corollary 1.5.15 the morphism $X \to X/G$ is finite. It follows from Proposition 1.5.13 that the morphism $X \to X/G$ satisfies the equivalent conditions of Lemma 3.3.1.

Finite group actions and qfh-sheaves

The following Lemma is closely related to [SV00, Lem. 5.16] and it tells us that if $\pi : X \to X/G$ is the quotient of a finite group acting admissibly on X then for any qfh-sheaf $\mathcal{F}$ the sections of $\mathcal{F}(X/G)$ are identified with the G-invariant sections of $\mathcal{F}(X)$.

Lemma 3.3.9. *Let $\pi : X \to S$ be a finite surjective morphism of schemes with S Noetherian and suppose that G is a finite group acting on S-automorphisms of X. Suppose further that (π, G) satisfies the equivalent conditions of Lemma 3.3.1. Let $\mathcal{F}$ be a qfh-sheaf. Then $\pi^* : \mathcal{F}(S) \to \mathcal{F}(X)$ induces an isomorphism $\mathcal{F}(S) \cong \mathcal{F}(X)^G$ and we have a factorization*

$$\begin{tikzcd}
\mathcal{F}(S) \arrow[rr, "\pi^*"] \arrow[dr] & & \mathcal{F}(X) \\
& \mathcal{F}(X)^G \arrow[ur, hook] &
\end{tikzcd}$$

Proof. By Corollary 3.3.4 we have that $\psi_{X/S} : G_X \to X \times_S X$ is a qfh-covering. Now for each $\sigma \in G$ we have a commutative diagram

$$\begin{tikzcd}
X \arrow[d, hook] \arrow[drr, bend left, "id_X"] \arrow[ddd, bend right=60, "\sigma"'] & & \\
G_{X/S} \arrow[d, "\psi_{X/S}"] & & \\
X \times_S X \arrow[rr, "pr_1"] \arrow[d, "pr_2"] & & X \arrow[d, "\pi"] \\
X \arrow[rr, "\pi"] & & S
\end{tikzcd}$$

From this we see that if $f \in \mathcal{F}(X)$ then if

$$pr_1^*(f) = pr_2^*(f)$$

we have that

$$f = \sigma^*(f)$$

for all σ. Conversely if

$$f = \sigma^*(f)$$

for all σ, then using that $\{X \to G_{X/S} = \coprod_{\sigma \in G} X\}_{\sigma \in G}$ is a qfh-covering we get that we must have that

$$\psi_{X/S}^* \circ pr_2^*(f) = \psi_{X/S}^* \circ pr_1^*(f)$$

and since $\psi_{X/S}$ is also a qfh-covering we must then have $pr_1^*(f) = pr_2^*(f)$. $\square$

3.4 An overview of the literature

This chapter closely follows and occasionally extends theory found in [Voe96],[SV96] and [SV00]. The following table explains how many of our statements compare to those found in the literature:

Comparison table			
Statement	Reference(s)	Statement comparison	Proof
Proposition 3.1.8	[Voe96, Proposition 3.1.3]	Identical	Identical
Corollary 3.1.9	[Voe96, Rmk after 3.1.3]	Identical	Expands
Proposition 3.1.10	[Voe96, Prop. 3.1.4]	Extends	Similar
Proposition 3.1.12	[SV96, Lem.10.3]	Generalises	Applies ideas from the original proof
Theorem 3.1.16	[Voe96, Thm.3.1.9]	Generalises	Similar
Lemma 3.2.2	[SV00, Lem.4.2.3]	Identical	Added
Proposition 3.2.5	[SV00, Prop.4.2.6]	Similar	A little different
Corollary 3.2.6	[SV00, Prop.4.2.6]	Similar	A little different
Theorem 3.2.9	[SV00, Thm.4.2.9]	Extends	Similar
Lemma 3.3.1	[SV96, Lemma 5.1]	Identical	Added
Corollary 3.3.3	[SV96, Corollary 5.2]	Identical	Added
Lemma 3.3.9	[SV00, Lem. 5.16]	Generalises	Central idea is the same. More details added

Chapter 4

Generalized seminormalization with applications

Inspired by the work of Ross [Ros10] and Huber-Kelly [HK18] we introduce in this chapter a "pointwise" construction suitable for the parallel study of semi and weak normalization in the context of rings and schemes. This is demonstrated in several instances: we prove a faithfully flat descent result for our construction yielding a new proof of the fact that semi and weak normality descends by faithfull flatness (Proposition 4.1.26) and in Lemma 4.2.22 we state and prove an analogue of Manaresi's characterisation of the weak normalization ([Man80, Thm. (I.6)]) yielding a new proof of Manaresi's Theorem in the case of rings with finitely many minimal prime ideals. Furthermore we apply our construction to study representable sheaves in the h-topologies, which extends Huber-Kelly's description of representable sheaves in several of the decomposable cousins of the h-topology ([HK18, Prop. 6.14]) to all Noetherian schemes and simultaneously gives a special case of Rydh's h-theoretic version ([Ryd10, Thm. (8.16)]).

We then finish the chapter by applying our efforts to give simple proofs of many of the results of Section 3.2 of [Voe96] together with some analogues for the decomposable cousins of the h topology.

Since Rydh has mentioned that one can make appropriate modifications to his methodology regarding [Ryd10, Thm. 8.16] (or to the original methods considered by Voevodsky in [Voe96]) to also cover the cases of decomposable topologies, and as some of the other results obtained are already known in some form, we stress that the novelty of this chapter mainly lies in our proofs and methodology.

Much of the theory appearing in this chapter was developed in collaboration with Jarle Stavnes.

4.1 The η-pointwise construction

The construction and its properties

Throughout the rest of this chapter $\eta : Id_{Fields} \to F$ will always denote a natural transformation from the identity functor on the category of fields to some endofunctor F such that the following conditions are satisfied

1. The field extension $F(\eta(K)) : F(K) \to F(F(K))$ is an isomorphism for all fields K.

2. The equivalent conditions of Lemma 1.4.25 are satisfied, i.e. $\eta(K)$ is a purely inseparable extension for all fields K.

Example 4.1.1. The two main examples of interests is the perfect closure ($\eta(K) : K \to K^{Perf}$) or the identity ($\eta(K) : K \overset{id_K}{\to} K$). There are also other examples of such natural transformations. Indeed for a prime number p and natural number n let $F_{p,n}$ be the endofunctor on the category of fields defined as follows: For a field K we set

$$F_{p,n}(K) = \begin{cases} K, & \text{if } \operatorname{char} K = p \text{ and } \operatorname{tr.deg}_{\mathbb{F}_P}(K) \leq n, \text{ or } \operatorname{char} K \neq p\,; \\ K^{Perf}, & \text{otherwise.} \end{cases}$$

Note that we have an obvious map $K \to F_{p,n}(K)$ which is natural in K.

Construction 4.1.2. For a ring A set

$$\mathrm{N}_\eta(A) := \prod_{p \in \operatorname{Spec}(A)} F(k(p)) \tag{4.1.1}$$

and let $t_A : A \to \mathrm{N}_\eta(A)$ be the induced homomorphism such that

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{N}_\eta(A) \\ \downarrow & & \downarrow \\ k(p) & \xrightarrow{\eta(k(p))} & F(k(p)) \end{array} \tag{4.1.2}$$

commutes for all $p \in \operatorname{Spec}(A)$. For a homomorphism of rings $f : A \to B$ let $\mathrm{N}_\eta(f) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(B)$ be the morphism such that

$$\begin{array}{ccc} \mathrm{N}_\eta(A) & \longrightarrow & \mathrm{N}_\eta(B) \\ \downarrow & & \downarrow \\ F(k(p)) & \xrightarrow{F(\overline{f}_{q/p})} & F(k(q)) \end{array} \tag{4.1.3}$$

commutes for all $q \in Spec(B), p \in Spec(A)$ with $f^{-1}(q) = p$. This makes N_η into an endofunctor on the category of rings and we have an obvious natural transformation $Id_{Rings} \to \mathrm{N}_\eta$. Let $\mathcal{P}$ be a subcategory of the category of rings. Let $\mathrm{M}_\eta^{\mathcal{P}}(A)$ be the subset of $\mathrm{N}_\eta(A)$ mapped to $\operatorname{im} t_R$ by $\mathrm{N}_\eta(f)$ for all homomorphisms $f : A \to R$ with $R \in \mathcal{P}$. Then $\mathrm{M}_\eta^{\mathcal{P}}(A)$ is a ring, and there is an induced homomorphism of rings $q_A : A \to \mathrm{M}_\eta^{\mathcal{P}}(A)$ natural in A which is an injection if and only if A is reduced (indeed, the kernel is the nilradical of A). Note that q_R is an isomorphism for all reduced rings $R \in \mathcal{P}$.

Remark 4.1.3. The main result of [Ros10] is that if A is a Noetherian ring and $\mathcal{P}$ the full subcategory of discrete valuation rings then $q_A : A \to \mathrm{M}_{Id}^{\mathcal{P}}(A)$ is the seminormalization of the ring A. Furthermore if one instead lets $\mathcal{P}$ denote the full subcategory of all valuation rings then combining Lemma 3.7 and Proposition 6.2 of [HK18] then if A is a ring with finitely many minimal prime ideals the map $q_A : A \to \mathrm{M}_{Id}^{\mathcal{P}}(A)$ is the seminormalisation of A. This will also follow from the main theorem of this section (Theorem 4.1.4).

Throughout V_η will denote the category of valuation rings R such that $\eta(R_{(0)}) : R_{(0)} \to F(R_{(0)})$ is an isomorphism. We will always write $\mathrm{M}_\eta(A)$ for the ring $\mathrm{M}_\eta^{V_\eta}(A)$. Note that if $f : A \to B$ is a morphism of two rings such that $\mathrm{N}_\eta(f) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(B)$ is an isomorphism then the induced map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ must necessarily give a bijection of sets, thus if in addition $\mathrm{Spec}(f)$ is a closed map of topological spaces which in particular is the case if f is integral then $\mathrm{Spec}(f)$ is a (universal) homeomorphism. The purpose of this section is to prove the following Theorem:

Theorem 4.1.4. *Let A be a ring with finitely many minimal prime ideals. Then*

1. *The map $q_A : A \to \mathrm{M}_\eta(A)$ is integral .*
2. *the induced map $\mathrm{Spec}(q_A) : \mathrm{Spec}(\mathrm{M}_\eta(A)) \to \mathrm{Spec}(A)$ is a bijection of sets.*
3. *For $q \in \mathrm{Spec}(\mathrm{M}_\eta(A))$ lying over $p \in \mathrm{Spec}(A)$ the field extension $\eta(k(p)) : k(p) \to F(k(p))$ factors through the map of residue fields $\overline{q_A}_{q/p} : k(p) \to k(q)$. In particular the map $\mathrm{Spec}(q_A)$ is a universal homeomorphism.*
4. *The map $\mathrm{N}_\eta(q_A) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(\mathrm{M}_\eta(A))$ is an isomorphism.*
5. *If $f : \mathrm{M}_\eta(A) \to B$ is any integral ring homomorphism to a reduced ring B such that $\mathrm{N}_\eta(f)$ is an isomorphism then f is an isomorphism.*
6. *If $g : A \to B$ is an integral ring homomorhism with $\mathrm{N}_\eta(g)$ an isomorphism then there exists a unique map $g' : B \to \mathrm{M}_\eta(A)$ such that $q_A = g' \circ g$.*

In particular if η is the identity ($\eta(K) : K \xrightarrow{Id_K} K$) then $q_A : A \to \mathrm{M}_{Id}(A)$ is isomorphic to the seminormalization of A as defined in for instance [Tra70]. Furthermore if η is the perfect closure ($\eta(K) : K \to K^{Perf}$) the map $q_A : A \to \mathrm{M}_{(-)^{Perf}}(A)$ is the absolute weak normalization of A introduced in [Ryd10] .

The theorem will be proved in many steps starting with the following Lemma:

Lemma 4.1.5. *If K is a field then* $\mathrm{M}_\eta(K) = F(K)$.

Proof. Let $x \in F(K)$ and let q be a positive power of $\operatorname{char} K$ such that $x^q \in K$ or in other words there is some $c \in K$ such that $x^q - c = 0$. Then for any map $f : K \to R$ with $R \in V_\eta$ we have that for any point $p \in \operatorname{Spec}(R)$ the image of x under the induced map $F(K) \to F(k(p))$ is a root of a polynomial of the form $T^q - f(c)$. Since this holds in particular for the zero ideal of R and R is integrally closed in its field of fractions we have some $r \in R$ satisfying $r^q = f(c)$, thus the image of x in $F(k(p))$ must coincide with the image of r for all $p \in \operatorname{Spec}(R)$. $\square$

Lemma 4.1.6. *Let A be an integral domain, and let K be its field of fractions. Suppose that $\eta(K) : K \to F(K)$ is an isomorphism. Then:*

1. *The canonical map $\mathrm{M}_\eta(A) \to F(K)$ is injective.*
2. *The inclusion $A \subseteq K \subseteq F(K)$ factors as $A \subseteq \mathrm{M}_\eta(A) \subseteq F(K)$.*

Proof. Consider the induced morphism $\mathrm{M}_\eta(A) \to \mathrm{M}_\eta(K) = F(K)$. Suppose that $(f_p)_p$ maps to 0. Then $f_{(0)} = 0$. Consider any nontrivial specialization $(0) \subset p$ in A and let $g : A \to R$ be a map to a valuation ring covering this specialization (Lemma 1.3.4) with $R_{(0)} = K$. Consider the induced ring homomorphism $\mathrm{M}_\eta(g) : \mathrm{M}_\eta(A) \to \mathrm{M}_\eta(R)$. The image $\mathrm{M}_\eta(g)((f_p)_p) \in \mathrm{M}_\eta(R)$ is contained in $\operatorname{im} t_R$, which means that there exists an element $f_R \in R$ such that $t_R(f_R) = \mathrm{M}_\eta(g)((f_p)_p)$. Since $f_{(0)} = 0$ it follows that $f_R = 0$ which implies that $f_p = 0$. Since p was arbitrary we obtain $f_p = 0$ for all p, thus $(f_p)_p = 0 \in \mathrm{M}_\eta(A)$. This proves (1) which again implies (2). $\square$

Proposition 4.1.7. *Let A be a integrally closed domain with field of fractions K. Suppose that $\eta(K) : K \to F(K)$ is an isomorphism. Then $q_A : A \to \mathrm{M}_\eta(A)$ is an isomorphism.*

Proof. We have that A is the intersection of the valuation rings of K containing A ([Stacks, Tag 090P]). Consider the induced factorizations $A \to \mathrm{M}_\eta(A) \to R \subseteq K$ for valuation rings R of K containing A. It follows from Lemma 4.1.6 that the homomorphism $\mathrm{M}_\eta(A) \to R$ is injective. By taking the intersection of all such R's, we get that $A = \mathrm{M}_\eta(A)$ as subrings of K. The result now follows. $\square$

Lemma 4.1.8. *Let A, B be rings and p_A, p_B denote the projections from the product onto A, B respectively. Then the induced map*

$$\mathrm{M}_\eta(A \times B) \xrightarrow{(\mathrm{M}_\eta(p_A), \mathrm{M}_\eta(p_B))} \mathrm{M}_\eta(A) \times \mathrm{M}_\eta(B) \tag{4.1.4}$$

is an isomorphism.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{M}_\eta(A \times B) \arrow[r, "(\mathrm{M}_\eta(p_A),\mathrm{M}_\eta(p_B))"] \arrow[d, hook] & \mathrm{M}_\eta(A) \times \mathrm{M}_\eta(B) \arrow[d, hook] \\ \mathrm{N}_\eta(A \times B) \arrow[r, "(\mathrm{N}_\eta(p_A),\mathrm{N}_\eta(p_B))"] & \mathrm{N}_\eta(A) \times \mathrm{N}_\eta(B) \end{tikzcd} \tag{4.1.5}$$

where the lower horizontal arrow is an isomorphism. Which immediately implies injectivity of the top horizontal morphism. For surjectivity let $(f_p)_p \in \mathrm{N}_\eta(A \times B)$ correspond to an element in the image of the right most vertical arrow. If $f : A \times B \to R$ is any map with $R \in V_\eta$, then since R is an integral domain it follows that f either factors through p_A or p_B hence it follows that $(f_p)_p \in \mathrm{M}_\eta(A \times B)$.

□

Proposition 4.1.9. *Let A be a normal ring with finitely many minimal prime ideals. Suppose that for every minimal prime ideal p of A the morphism $\eta(k(p))$ is an isomorphism. Then $q_A : A \to \mathrm{M}_\eta(A)$ is an isomorphism.*

Proof. If A is a normal ring with finitely many minimal prime ideals then there exists finitely many normal domains $A_1, \ldots, A_n$ such that $A \cong \prod_i A_i$ ([Stacks, Tag 030C]). Thus by Lemma 4.1.8 and Proposition 4.1.7 we have

$$\mathrm{M}_\eta(A) \cong \prod \mathrm{M}_\eta(A_i) \cong \prod A_i \cong A. \tag{4.1.6}$$

□

Proposition 4.1.10. *Let A be a ring with finitely many minimal prime ideals. Then*

1. *The map $q_A : A \to \mathrm{M}_\eta(A)$ is integral .*
2. *the induced map $\mathrm{Spec}(q_A) : \mathrm{Spec}(\mathrm{M}_\eta(A)) \to \mathrm{Spec}(A)$ is a bijection of sets.*
3. *For $q \in \mathrm{Spec}(\mathrm{M}_\eta(A))$ lying over $p \in \mathrm{Spec}(A)$ the field extension $\eta(k(p)) : k(p) \to F(k(p))$ factors through the map of residue fields $\overline{q_A}_{q/p} : k(p) \to k(q)$.*
4. *The map $\mathrm{N}_\eta(q_A) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(\mathrm{M}_\eta(A))$ is an isomorphism.*

Proof. Let $\tilde{A}$ denote the integral closure of A_{red} in the product $\prod_{\substack{p \in \mathrm{Spec}(A), \\ p \text{ minimal}}} F(k(p))$. The ring $\tilde{A}$ satisfies the properties of Proposition 4.1.9. Consider now the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "q_A"] & \tilde{A} \arrow[d, "q_{\tilde{A}}"', "\cong"] \\ \mathrm{M}_\eta(A) \arrow[r, "\mathrm{M}_\eta(i)"] & \mathrm{M}_\eta(\tilde{A}). \end{tikzcd} \tag{4.1.7}$$

Since i is integral it follows that q_A must also be integral proving (1). For (2) let $f = (f_p)_p \in \mathrm{M}_\eta(A)$ and denote its image in $\tilde{A}$ by $\tilde{f}$. This element has the following property: For any $q \in \mathrm{Spec}(\tilde{A})$ with $q \cap A = p$ the image of $\tilde{f}$ under the map $\tilde{A} \to k(q) \overset{\eta(k(q))}{\to} F(k(q))$ coincides with the image of f_p under $F(\bar{i}_{q/p}) : F(k(p)) \to F(k(q))$. This shows that for fixed $p \in \mathrm{Spec}(A)$ if $f_p \neq 0$ then $\tilde{f}$ cannot be contained in any prime ideal $q \in \mathrm{Spec}(\tilde{A})$ lying over p, and if $f_p = 0$ we must necessarily have $\tilde{f} \in \mathfrak{q}$ for every $\mathfrak{q} \in \mathrm{Spec}(\tilde{A})$ with q lying over p. Hence if we have two prime ideals p_1, p_2 of $\mathrm{M}_\eta(A)$ lying over p then surjectivity of $\mathrm{Spec}(\tilde{A}) \to \mathrm{Spec}(\mathrm{M}_\eta(A))$ implies that any element of p_1 must be contained in p_2 and vice versa hence (2) follows. Part (3) is now a consequence of the fact that (2) implies that any prime ideal of $\mathrm{M}_\eta(A)$ is the preimage of a prime ideal of $\mathrm{N}_\eta(A)$. Finally part (4) follows from part (3) and the condition that $F \to F \circ F$ is an isomorphism. $\square$

The proposition easily gives the following consequences:

Corollary 4.1.11. *Let $f : A \to B$ be a morphism of rings both of which have finitely many minimal prime ideals. Then the following is satisfied:*

1. *The map f is integral if and only if the induced map $\mathrm{M}_\eta(f)$ is.*

2. *The map $\mathrm{Spec}(f)$ is surjective if and only if $\mathrm{Spec}(\mathrm{M}_\eta(f))$ is.*

   Suppose furthermore that the induced map $\mathrm{M}_\eta(f) : \mathrm{M}_\eta(A) \to \mathrm{M}_\eta(B)$ is an isomorphism. Then we also have:

3. *The map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives a bijection of prime ideals of A and B.*

4. *For any $q \in \mathrm{Spec}(B)$ with $p = f^{-1}(q)$ the map $\eta(k(p)) \to F(k(p))$ factors through the induced map of residue fields $\overline{f}_{q/p} : k(p) \to k(q)$.*

Recalling Lemma 1.4.28 and Lemma 1.4.30 we also obtain:

Corollary 4.1.12. *Let $f : A \to B$ be a morphism of rings both of which have finitely many minimal prime ideals. Then the induced map $\mathrm{Spec}(f)$ is universally injective (resp. a universal homeomorphism) if and only if $\mathrm{Spec}(\mathrm{M}_\eta(f))$ is.*

Lemma 4.1.13. *Let $f : A \to B$ be an integral homomorphism such that $\mathrm{N}_\eta(f) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(B)$ is an isomorphism. Then any homomorphism $g : A \to R$ with $R \in V_\eta$ factors through f.*

Proof. Let $g : A \to R$ be such a homomorphism and let $p = g^{-1}((0)) \subset A$. Since $\mathrm{N}_\eta(f)$ is an isomorphism by assumption there is a unique prime ideal

$p' \subset B$ such that $f^{-1}(p') = p$ and further $F(\overline{f}_{p'/p}) : F(k(p)) \to F(k(p'))$ is an isomorphism. From this we obtain a commutative diagram of the form

$$\begin{tikzcd} B \arrow[r] & R_{(0)} \\ A \arrow[u, "f"] \arrow[r, "g"] & R \arrow[u] \end{tikzcd} \tag{4.1.8}$$

Now apply 1.3.11 to conclude. □

Corollary 4.1.14. *Let A be a ring. Suppose that $f : A \to B$ is any integral morphism such that $\mathrm{N}_\eta(f)$ is an isomorphism. Then $\mathrm{M}_\eta(f)$ is an isomorphism.*

Proof. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathrm{M}_\eta(A) \arrow[r] \arrow[d, hook] & \mathrm{M}_\eta(B) \arrow[d, hook] \\ \mathrm{N}_\eta(A) \arrow[r, "\cong"] & \mathrm{N}_\eta(B) \end{tikzcd} \tag{4.1.9}$$

from which injectivity of $\mathrm{M}_\eta(A) \to \mathrm{M}_\eta(B)$ is clear. By Lemma 4.1.13 surjectivity is also clear. □

Corollary 4.1.15. *Let R be a valuation ring with field of fractions K. The ring $\mathrm{M}_\eta(R)$ is isomorphic to the integral closure of R in $F(K)$. Furthermore $\mathrm{M}_\eta(R)$ is a valuation ring with field of fractions $F(K)$ and the image of $\mathrm{M}_\eta(R)$ in $F(K)$ intersected with K coincides with the image of R in $F(K)$.*

Proof. Let $\tilde{R}$ denote the integral closure of R in $F(K)$ and note that since the extension $\eta(K)$ is purely inseparable and therefore algebraic the field of fractions of $\tilde{R}$ is $F(K)$. If p is the exponential characteristic of the field K then for a given $x \in F(K)$ have some power of p denoted by q such that $x^q \in K$. By Lemma 1.3.3 we then either have $x^q \in R$ which necessarily means that $x \in \tilde{R}$ or we have $(x^q)^{-1} = (x^{-1})^q \in R$; thus we see that $\tilde{R}$ is a valuation ring.

Hence for every $x \in \tilde{R}$ there is some positive integer n such that $x^{p^n} \in R$ (and $p^n x = 0 \in R$). Thus by [Stacks, Tag 0BRA] it follows that the induced map $\mathrm{N}_\eta(R) \to \mathrm{N}_\eta(\tilde{R})$ is an isomorphism, thus by Corollary 4.1.14 the map

$$\mathrm{M}_\eta(R) \to \mathrm{M}_\eta(\tilde{R}) \tag{4.1.10}$$

is an isomorphism. Proposition 4.1.7 now allows us to conclude the proof. □

Lemma 4.1.16. *Let A be a ring with finitely many minimal prime ideals. Then the maps $q_{\mathrm{M}_\eta(A)} : \mathrm{M}_\eta(A) \to \mathrm{M}_\eta(\mathrm{M}_\eta(A))$ and $\mathrm{M}_\eta(q_A) : \mathrm{M}_\eta(A) \to \mathrm{M}_\eta(\mathrm{M}_\eta(A))$ coincide. Moreover this map is an isomorphism.*

Proof. For a given $p \in \mathrm{Spec}(A)$ we have by Proposition 4.1.10 a unique $q \in \mathrm{Spec}(\mathrm{M}_\eta(A))$ lying over p and moreover the map $\eta(k(p)) : k(p) \to F(k(p))$ factors through the induced map of residue fields $\overline{q_A}_{q/p} : k(p) \to k(q)$. Thus since $k(q)/k(p)$ is purely inseparable it follows that the following diagram is commutative

$$\begin{tikzcd} \mathrm{M}_\eta(A) \arrow[r] \arrow[d] & k(q) \arrow[dd, "\eta(k(q))"] \\ \mathrm{N}_\eta(A) \arrow[d, "pr_p"] & \\ F(k(p)) \arrow[r, "F(\overline{q_A}_{q/p})"] & F(k(q)). \end{tikzcd} \tag{4.1.11}$$

Hence by the definition of $q_{\mathrm{M}_\eta(A)}$ and $\mathrm{M}_\eta(q_A)$ we see that they must coincide. The last statement follows from Corollary 4.1.14 and Proposition 4.1.10 again. ∎

Proposition 4.1.17. *Let A be a ring with finitely many minimal prime ideals. Then the following hold true:*

1. *If $f : \mathrm{M}_\eta(A) \to B$ is any integral ring homomorphism to a reduced ring B such that $\mathrm{N}_\eta(f)$ is an isomorphism then f is an isomorphism.*
2. *If $g : A \to B$ is an integral ring homomorhism with $\mathrm{N}_\eta(g)$ an isomorphism then there exists a unique map $g' : B \to \mathrm{M}_\eta(A)$ such that $q_A = g' \circ g$.*

Proof. **For (1):** Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{M}_\eta(A) \arrow[r, "f"] \arrow[d, "q_{\mathrm{M}_\eta(A)}"] & B \arrow[d, hook, "q_B"] \\ \mathrm{M}_\eta(\mathrm{M}_\eta(A)) \arrow[r, "\mathrm{M}_\eta(f)"] & \mathrm{M}_\eta(B). \end{tikzcd} \tag{4.1.12}$$

By Lemma 4.1.16 the leftmost vertical arrow is an isomorphism and by Corollary Corollary 4.1.14 the lower horizontal arrow is also an isomorphism hence we conclude that f must necessarily be an isomorphism.

For (2): Suppose first that there exists a map $g' : B \to \mathrm{M}_\eta(A)$ such that $q_A = g' \circ g$. Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "q_A"] & B \arrow[r, "g'"] \arrow[d, "q_B"] & \mathrm{M}_\eta(A) \arrow[d, "q_{\mathrm{M}_\eta(A)}"] \\ \mathrm{M}_\eta(A) \arrow[r, "\mathrm{M}_\eta(g)"] & \mathrm{M}_\eta(B) \arrow[r, "\mathrm{M}_\eta(g')"] & \mathrm{M}_\eta(\mathrm{M}_\eta(A)). \end{tikzcd} \tag{4.1.13}$$

From which we deduce that

$$q_{\mathrm{M}_\eta(A)} \circ g' = \mathrm{M}_\eta(q_A) \circ \mathrm{M}_\eta(g)^{-1} \circ q_B = q_{\mathrm{M}_\eta(A)} \circ \mathrm{M}_\eta(g)^{-1} \circ q_B. \tag{4.1.14}$$

Hence in this case we must have $g' = \mathrm{M}_\eta(g)^{-1} \circ q_B$ and since the right hand side always exists we have proved both the existence and uniqueness of g'. $\square$

We can now finally prove the main theorem of this section:

Proof of Theorem 4.1.4. Parts (1) through (4) are proved in Proposition 4.1.10 and (5)-(6) are both proved in Proposition 4.1.17. $\square$

The relative case and weak normalization

Item 6 of Theorem 4.1.4 gives a defining universal property of the map $q_A : A \to \mathrm{M}_\eta(A)$. For a fixed ring map $A \to B$ we can apply the pointwise η-construction to prove the existence of a ring with a similar universal property relative to the morphism $A \to B$.

Construction 4.1.18. Let A be a ring with finitely many minimal prime ideals and $f : A \to B$ a morphism such that $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Let $(B/A)_\eta$ be the pullback of the two maps $\mathrm{M}_\eta(f) : \mathrm{M}_\eta(A) \to \mathrm{M}_\eta(B)$ and $q_B : B \to \mathrm{M}_\eta(B)$. Denote the two projections by $u_{B/A} : (B/A)_\eta \to \mathrm{M}_\eta(A)$ and $i_{B/A} : (B/A)_\eta \to B$. Note that $i_{B/A}$ is injective. Finally let $q_{B/A} : A \to (B/A)_\eta$ be the unique map determined by q_A and f.

Proposition 4.1.19. *Let $A \to B$ be a map of rings where A has finitely many minimal prime ideals and the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Then the following assertions hold true:*

1. *The homomorphism $q_{B/A} : A \to (B/A)_\eta$ is integral.*
2. *The induced map $\mathrm{Spec}(q_{B/A}) : \mathrm{Spec}((B/A)_\eta) \to \mathrm{Spec}(A)$ is a bijection of sets.*
3. *For $q \in \mathrm{Spec}((B/A)_\eta)$ lying over $p \in \mathrm{Spec}(A)$ the field extension $\eta(k(p)) : k(p) \to F(k(p))$ factors through the map of residue fields $\overline{q_{B/A}}_{q/p} : k(p) \to k(q)$. In particular the map $\mathrm{Spec}(q_{B/A})$ is a universal homeomorphism.*
4. *The map $\mathrm{N}_\eta(q_{B/A}) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta((B/A)_\eta)$ is an isomorphism.*
5. *For any factorization of f of the form*

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[rr, bend left, "f"] & B' \arrow[r, "f_2"] & B \end{tikzcd} \tag{4.1.15}$$

   where f_1 is integral and $\mathrm{N}_\eta(f_1)$ an isomorphism the map f_2 factors uniquely through $i_{B/A} : (B/A)_\eta \to B$.

   In particular if η is the identity the map (resp. the perfect closure) $q_{B/A}$ is the semi-normalization (resp. weak normalization) of A in B.

Proof. Since Theorem 4.1.4 tells us that q_A is integral and the map q_B has nil-potent kernel it follows easily that $q_{B/A}$ is integral and that $u_{B/A}$ is integral with nilpotent kernel. Hence $\mathrm{Spec}(u_{B/A})$ is surjective from which we easily deduce that $\mathrm{Spec}(q_{B/A})$ must be bijective. Thus (1) and (2) are proved. Parts (3) and (4) immediately follow from Theorem 4.1.4 again and the standing assumptions on η. For the last part recall from Corollary 4.1.14 that $\mathrm{M}_\eta(f_1)$ is an isomorphism thus the maps $\mathrm{M}_\eta(f_1)^{-1} \circ q_{B'}$ and f_2 induce a map $B' \to (B/A)_\eta$ with the desired property and the uniqueness of this map follows from the fact that $\mathrm{M}_\eta(f)$ is a monomorphism and part (2) of Proposition 4.1.17. □

Lemma 4.1.20. *Suppose that A is a ring with finitely many minimal prime ideals and $f : A \to B$ is a ring homomorphism with* $\mathrm{Spec}(f)$ *surjective. Then letting C denote the integral closure of A in B the induced map*

$$j : (C/A)_\eta \to (B/A)_\eta \tag{4.1.16}$$

is an isomorphism.

Proof. Recall from Proposition 4.1.19 that the map $q_{B/A}$ is integral, hence the inclusion $i_{A/B} : (B/A)_\eta \to B$ factors through C. Moreover since $q_{B/A} = j \circ q_{C/A}$ it follows from part (5) of Proposition 4.1.19 that there is a unique map $(B/A)_\eta \to (C/A)_\eta$ which is easily seen to be an inverse to j. □

Taking η to be the identity (resp. perfect closure) in the following Lemma gives us Lemma 1.2 of [Tra70] (resp. Corollary (I.8) of [Man80]) for rings with finitely many minimal prime ideals.

Lemma 4.1.21. *Let A be a ring with finitely many minimal prime ideals and suppose we have maps $f : A \to B$ and $g : B \to C$. If $q_{B/A}$ and $q_{C/B}$ are isomorphisms then so is $q_{C/A}$. Moreover if $g : B \to C$ is injective then we have that if $q_{C/A}$ is an isomorphism then so is $q_{B/A}$*

Proof. Consider the following diagram where each square is a pullback:

$$\begin{tikzcd}
(B/A)_\eta \arrow[r] \arrow[d, "q'"] & B \arrow[d, "q_{C/B}"] & \\
(C/A)_\eta \arrow[r] \arrow[d] & (C/B)_\eta \arrow[r] \arrow[d] & C \arrow[d, "q_C"] \\
\mathrm{M}_\eta(A) \arrow[r, "\mathrm{M}_\eta(f)"] & \mathrm{M}_\eta(B) \arrow[r, "\mathrm{M}_\eta(g)"] & \mathrm{M}_\eta(C).
\end{tikzcd} \tag{4.1.17}$$

We see that if $q_{C/B}$ is an isomorphism then so is $q' : (B/A)_\eta \to (C/A)_\eta$ and since $q_{C/A} = q' \circ q_{B/A}$ the first statement follows. For the last note that if $q_{C/A}$ is an isomorphism then clearly $q_{B/A}$ is an injection and if g is an injection then so is q' hence $q_{B/A}$ must also be a surjection completing the proof. □

Relations to Kollar's description

The description of the semi/weak normalization given in the proof of [Kol96, Ch.1, Prop. 7.2.3] will be useful for us later on. Thus we will use our pointwise description to directly prove that for $F = Id$ (resp. $F = Perf(-)$) our $(B/A)_\eta$ coincides with Kollár's semi (resp. weak) normalization of A in B, at least when B is integral over A. In order to do so we will need the following lemma:

Lemma 4.1.22. *Let $f : A \to B$ be an integral ring homomorphism with* $\mathrm{Spec}(f)$ *surjective. Then $x \in \mathrm{N}_\eta(A)$ is contained in $\mathrm{M}_\eta(A)$ if and only if $\mathrm{N}_\eta(f)(x) \in \mathrm{M}_\eta(B)$.*

Proof. One implication is obvious. For the other suppose $\mathrm{N}_\eta(f)(x) \in \mathrm{M}_\eta(B)$. Let $g : A \to R$ be any ring homomorphism with $R \in V_F$. By Lemma 1.3.12 there exists a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ R & \xrightarrow{h} & R' \end{array}$$

with R' a valuation ring, $\mathrm{Spec}(h)$ surjective and $R' \cap R_{(0)} = R$ (where the intersection is taken inside $R'_{(0)}$). By assumption there is some $r' \in R'$ with

$$t_{R'}(r') = \mathrm{N}_\eta(g')(\mathrm{N}_\eta(f)(f)(x)) = \mathrm{N}_\eta(h)(\mathrm{N}_\eta(g)(x))$$

from which we see that $r' \in R' \cap R_{(0)}$ thus $r' \in R$ and by injectivity of $\mathrm{N}_\eta(h)$ we must necessarily have $t_R(r') = \mathrm{N}_\eta(g)(x)$ thus $x \in \mathrm{M}_\eta(A)$ completing the proof. $\square$

Proposition 4.1.23. *Suppose that $f : A \to B$ is an integral ring homomorphism with* $\mathrm{Spec}(f)$ *surjective. Then the following diagram*

$$\begin{array}{ccc} (B/A)_\eta & \xrightarrow{i_{B/A}} & B \\ \downarrow{\scriptstyle u_{B/A}} & & \downarrow{\scriptstyle q_B} \\ \mathrm{M}_\eta(A) & \xrightarrow{\mathrm{M}_\eta(f)} & \mathrm{M}_\eta(B) \\ \downarrow & & \downarrow \\ \mathrm{N}_\eta(A) & \xrightarrow{\mathrm{N}_\eta(f)} & \mathrm{N}_\eta(B) \end{array}$$

is a pullback of $t_B : B \to \mathrm{N}_\eta(B)$ along $\mathrm{N}_\eta(f) : \mathrm{N}_\eta(A) \to \mathrm{N}_\eta(B)$.

Proof. Follows easily from Lemma 4.1.22. $\square$

From this proposition it follows that the image of $i_{B/A}$ in B is exactly the description of the semi/weak normalization given in [Kol96].

The following example is the standard example telling us that the notions of semi and weak normality don't necessarily coincide in positive characteristic.

Example 4.1.24. Consider the Whitney umbrella with coordinate ring $A = k[X,Y,Z]/(XY^2 - Z^2)$ with normalization $k[S,T]$ given by $X \mapsto S^2, Y \mapsto T, Z \mapsto ST$ which induces a homeomorphism of spectra (at least when k is algebraically closed). Using Lemma 4.1.23 we then easily see that A is seminormal, and if k has characteristic two then the weak normalization of A coincides with the normalization.

Faithfully flat descent

The following is an analogue of [Yan83, Sec.2, Prop.1]

Lemma 4.1.25. *Suppose we have a pullback diagram of commutative rings:*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"] & B \arrow[d, "g"] \\ A' \arrow[r, "j'"] & B' \end{tikzcd}$$

with $\operatorname{Spec}(j')$ *and* $\operatorname{Spec}(j)$ *surjective. If the map* $q_{B'/A'} : A' \to (B'/A')_\eta$ *is an isomorphism then so is the map* $q_{B/A} : A \to (B/A)_\eta$*.*

Proof. Suppose that we are given maps $t_1 : D \to \mathrm{M}_\eta(A)$ and $t_2 : D \to B$ such that

$$\mathrm{M}_\eta(j) \circ t_1 = q_B \circ t_2.$$

Then by assumption we get a unique map $t' : D \to A'$ such that $q_{A'} \circ t' = \mathrm{M}_\eta(f) \circ t_1$ and $j' \circ t' = g \circ t_2$. From this latter equality we get induced a unique map $t : D \to A$ such that

$$f \circ t = t' \text{ and } j \circ t = t_2$$

Since the map $\mathrm{M}_\eta(j)$ is a monomorphism it is clear that we must have $q_A \circ t = t_1$. Furthermore since j' is necessarily a monomorphism j must also be a monomorphism hence t is the unique map satisfying the following two equalities:

$$j \circ t = t_2 \text{ and } q_A \circ t = t_1.$$

□

If η is the perfect closure then the following Proposition gives us [Man80, (II.1)] and [Yan83, Sec 2. , Cor.1] for rings with finitely many minimal prime ideals. Similarly if η is the identity then it gives us Theorem 1.6 of [GT80].

Proposition 4.1.26. *(Faithful flat descent) Let $f : A \to A'$ be a faitfully flat ring homomorphism of rings with finitely many minimal prime ideals and $j : A \to B$ a morphism with* Spec(j) *surjective. Set $B' := A' \otimes_A B$. If $q_{B'/A'} : A' \to (B'/A')_\eta$ is an isomorphism then so is $q_{B/A} : A \to (B/A)_\eta$.*

Proof. Note that if $q_{B'/A'}$ is an isomorphism then it follows that $A' \to B'$ must necessarily be an injection. Lemma 4.1.25 and Lemma 1.4.22 now let us conclude. □

The following is the analogue of Corollary (II.2) of [Man80] and Corollary 1.7 of [GT80].

Corollary 4.1.27. *Let $A \to A'$ be a faithfully flat ring homomorphism of rings with finitely many minimal prime ideals, and $\overline{A}$ (resp. $\overline{A'}$) denote the integral closure of A (resp. of A') in $Q(A)$ (resp. in $Q(A')$). If*

$$q_{\overline{A'}/A'} : A' \to (\overline{A'}/A')_\eta$$

is an isomorphism then so is

$$q_{\overline{A}/A} : A \to (\overline{A}/A)_\eta$$

In particular if A is a local Noetherian ring then $q_{\overline{A}/A}$ is an isomorphism if this is the case when A is replaced with its completion.

Proof. By Lemma 1.4.7 we have that A' and $A' \otimes_A \overline{A}$ have the same total ring of fractions hence we have an inclusion $A' \otimes_A \overline{A} \subset \overline{A'}$. From Lemma 4.1.21 it follows that if $q_{\overline{A}/A} : A \to (\overline{A}/A)_\eta$ is an isomorphism then so is $A' \to (\overline{A} \otimes_A A'/A')_\eta$. By Proposition 4.1.26 we are done. □

Study on distinguished opens

We will now begin to make some preparations for the scheme theoretic story to be considered in the next section.

Lemma 4.1.28. *Let A be a ring and $f \in A$ be an element of A. Then the natural map*

$$\mathrm{N}_\eta(A)_f \to \mathrm{N}_\eta(A_f)$$

is an isomorphism.

Proof. Surjectivity of the map is evident. For injectivity note if $x \in \mathrm{N}_\eta(A)_f$ is an element of the kernel then we must necessarily have $f \cdot x = 0$; hence $x = 0$. □

Lemma 4.1.29. *For a ring A with finitely many minimal prime ideals and an element f of A, let $\phi : A \to B$ be a map with* $\operatorname{Spec}(\phi)$ *surjective. Then the induced map*

$$((B/A)_\eta)_f \to (B_f/A_f)_\eta$$

is an isomorphism.

Proof. Let C denote the integral closure of A in B and consider the following commutative diagram

$$\begin{tikzcd} ((C/A)_\eta)_f \arrow[r] \arrow[d] & (C_f/A_f)_\eta \arrow[d] \\ ((B/A)_\eta)_f \arrow[r] & (B_f/A_f)_\eta. \end{tikzcd}$$

By Lemma 4.1.20 both vertical arrows are isomorphisms. Hence we can reduce to the case where ϕ is integral. By Proposition 4.1.23 the following sequence of A-modules is exact:

$$(B/A)_\eta \hookrightarrow (\mathrm{N}_\eta(A) \times B) \xrightarrow{\mathrm{N}_\eta(\phi)-t_B} \mathrm{N}_\eta(B)$$

Thus by exactness of localization and Lemma 4.1.28 we get an exact sequence of the form:

$$((B/A)_\eta)_f \hookrightarrow (\mathrm{N}_\eta(A_f) \times B_f) \xrightarrow{\mathrm{N}_\eta(\phi_f)-t_{B_f}} \mathrm{N}_\eta(B_f)$$

which completes the proof. $\square$

Corollary 4.1.30. *For a ring A with finitely many minimal prime ideals and an element $f \in A$ the induced map*

$$\mathrm{M}_\eta(A)_f \to \mathrm{M}_\eta(A_f)$$

is an isomorphism.

Proof. Let $\tilde{A}$ denote the normalization of A in $\prod_{\substack{p \in \operatorname{Spec}(A),\\ p \text{ minimal}}} F(k(p))$. Then from Lemma 4.1.29 the induced map

$$(\tilde{A}/A)_{\eta f} \to (\tilde{A}_f/A_f)_\eta$$

is an isomorphism and the desired result follows now easily from Proposition 4.1.9. $\square$

Lemma 4.1.31. *Let A be a ring with finitely many minimal prime ideals and $\phi : A \to B$ be a morphism of rings with* $\operatorname{Spec}(\phi)$ *surjective.*

1. *If $A \to (B/A)_\eta$ is an isomorphism and f is any element of A then*

$$A_f \to (B_f/A_f)_\eta$$

is an isomorphism.

2. *Suppose that $f_1, \ldots, f_n \in A$ generate the unit ideal $(1) = A$ and that the induced maps*

$$A_{f_i} \to (B_{f_i}/A_{f_i})_\eta \cong (B/A)_{\eta f_i}$$

are all isomorphisms. Then

$$q_{B/A} : A \to (B/A)_\eta$$

is an isomorphism.

Proof. The first assertion follows directly from 4.1.29 and part (2) is proved in a standard manner. □

4.2 The case of schemes

We use the notation of Section 4.1 throughout.

Construction 4.2.1. For a scheme X let $\mathscr{N}_\eta(X)$ be the Zariski sheaf on X given by $U \mapsto \mathscr{N}_\eta(X)(U) := \prod_{x \in U} F(k(x))$ and the obvious restriction maps. We also have a canonical map $t_X : \mathcal{O}_X \to \mathscr{N}_\eta(X)$ giving $\mathscr{N}_\eta(X)$ the structure of an $\mathcal{O}_X$-algebra. For any map $f : Y \to X$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & f_*\mathcal{O}_Y \arrow[d, "f_*(t_Y)"] \\ \mathscr{N}_\eta(X) \arrow[r, "\mathscr{N}_\eta(f)"] & f_*\mathscr{N}_\eta(Y). \end{tikzcd} \tag{4.2.1}$$

Here $\mathscr{N}_\eta(f)$ is the unique map such that for any open subset U of X with $y \in f^{-1}(U)$ the following diagram commutes:

$$\begin{tikzcd} \mathscr{N}_\eta(X)(U) \arrow[r, "\mathscr{N}_\eta(f)"] \arrow[d] & \mathscr{N}_\eta(Y)(f^{-1}(U)) \arrow[d] \\ k(f(y)) \arrow[r, "\overline{f}_{y/f(y)}"] & k(y). \end{tikzcd} \tag{4.2.2}$$

Note that if $\mathscr{N}_\eta(f)$ is an isomorphism then f has to give a bijection of underlying sets.

For a full subcategory $\mathcal{P}$ of the category of schemes we let $\mathscr{M}_\eta^{\mathcal{P}}(X)$ be the sub-presheaf of $\mathscr{N}_\eta(X)$ such that for an open subset U of X the set

$$\mathscr{M}_\eta^{\mathcal{P}}(X)(U) \subset \mathscr{N}_\eta(X)(U) \tag{4.2.3}$$

consists of exactly those elements $(f_x)_{x\in U} \in \mathscr{N}_\eta(X)(U)$ such that if $g : Y \to X$ is a morphism with $Y \in \mathcal{P}$ then

$$\mathscr{N}_\eta(g)(U)((f_x)_{x\in U}) \in \operatorname{im}(g_* t_Y(U)). \tag{4.2.4}$$

The map t_X factors through $\mathscr{M}_\eta^{\mathcal{P}}(X)$ and we let q_X denote this map $\mathcal{O}_X \to \mathscr{M}_\eta^{\mathcal{P}}(X)$. Furthermore if $f : Y \to X$ is a morphism then $\mathscr{N}_\eta(f)|_{\mathscr{M}_\eta^{\mathcal{P}}(X)}$ factors through $f_*\mathscr{M}_\eta^{\mathcal{P}}(Y)$ thus we get a morphism $\mathscr{M}_\eta^{\mathcal{P}}(f) : \mathscr{M}_\eta^{\mathcal{P}}(X) \to f_*\mathscr{M}_\eta^{\mathcal{P}}(Y)$. Note that if X is a reduced scheme with $X \in \mathcal{P}$ then q_X is an isomorphism.

Throughout we let $\mathscr{V}_\eta$ denote the full subcategory of schemes where objects are spectrums of valuation rings R with $\eta(R_{(0)})$ an isomorphism. We shall always denote $\mathscr{M}_\eta^{\mathscr{V}_\eta}$ by $\mathscr{M}_\eta$. Finally observe that if $g : Z \to Y$ and $f : Y \to X$ are morphisms of schemes then we have

$$\mathscr{N}_\eta(f \circ g) = f_*\mathscr{N}_\eta(g) \circ \mathscr{N}_\eta(f);$$
$$\mathscr{M}_\eta(f \circ g) = f_*\mathscr{M}_\eta(g) \circ \mathscr{M}_\eta(f).$$

Lemma 4.2.2. *Let X be a scheme where quasi-compact opens have finitely many irreducible components. The following statements hold true:*

1. *The presheaf $\mathscr{M}_\eta(X)$ is a Zariski sheaf.*
2. *The map $t_X : \mathcal{O}_X \to \mathscr{N}_\eta(X)$ makes $\mathscr{N}_\eta(X)$ into a quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that for any affine open* $\operatorname{Spec}(A)$ *of X we have*
$$\mathscr{N}_\eta(X)|_{\operatorname{Spec}(A)} \cong \widetilde{\mathrm{N}_\eta(A)}. \tag{4.2.5}$$
3. *The map $q_X : \mathcal{O}_X \to \mathscr{M}_\eta(X)$ makes $\mathscr{M}_\eta(X)$ into a quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that for any affine open* $\operatorname{Spec}(A)$ *of X we have*
$$\mathscr{M}_\eta(X)|_{\operatorname{Spec}(A)} \cong \widetilde{\mathrm{M}_\eta(A)}. \tag{4.2.6}$$

Proof. **For (1):** Note first that the presheaf $\mathscr{N}_\eta(X)$ is a Zariski sheaf. Hence to see that $\mathscr{M}_\eta(X)$ is a Zariski sheaf it is enough to show that if U is any open subset of X and $u = (u_x)_{x\in U}$ an element of $\mathscr{N}_\eta(X)(U)$ such that there is an open cover $\{U_i\}$ of U with $u|_{U_i} \in \mathscr{M}_\eta(X)(U_i)$ then we must necessarily have $u \in \mathscr{M}_\eta(X)(U)$. To see this just notice that if $g : \operatorname{Spec}(R) \to X$ is a morphism with $\operatorname{Spec}(R) \in \mathscr{V}_\eta$ then since $g_*t_{\operatorname{Spec}(R)}$ is injective and both $g_*\mathcal{O}_Y$ and $g_*\mathscr{N}_\eta(Y)$ are sheaves we see easily that the $\mathscr{N}_\eta(g)(u)$ is in the image of $g_*t_{\operatorname{Spec}(R)}$.

For (2): For any affine open subscheme $\operatorname{Spec}(A)$ of X we have an obvious isomorphism

$$\mathscr{N}_\eta(X)(\operatorname{Spec}(A)) \to \mathrm{N}_\eta(A) \tag{4.2.7}$$

and since the map $\mathrm{N}_\eta(A)_f \to \mathrm{N}_\eta(A_f)$ is an isomorphism for any $f \in A$ (Lemma 4.1.28) we can conclude.

For (3): We first claim that if $\mathrm{Spec}(A)$ is an affine open subscheme of X then the image of $\mathscr{M}_\eta(X)(\mathrm{Spec}(A))$ in $\mathrm{N}_\eta(A)$ under the canonical isomorphism $\mathscr{N}_\eta(X)(\mathrm{Spec}(A)) \to \mathrm{N}_\eta(A)$ is exactly the subring $\mathrm{M}_\eta(A)$. One inclusion is rather obvious so suppose that $u = (u_x)_{x\in\mathrm{Spec}(A)} \in \mathscr{N}_\eta(X)(\mathrm{Spec}(A))$ maps to an element of $\mathrm{M}_\eta(A)$. If $g : \mathrm{Spec}(R) \to X$ is any morphism with $\mathrm{Spec}(R) \in \mathscr{V}_\eta$ we must show that the element $\mathrm{N}_\eta(g)(u)$ is in the image of $t_{\mathrm{Spec}(R)}(g^{-1}(\mathrm{Spec}(A))$. Since we don't necessarily have $g^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(R)$ we cannot conclude straight away, but since $g^{-1}(\mathrm{Spec}(A))$ can be covered by affine opens which by Lemma 1.3.5 can be taken to be valuation rings with the same function field as R the claim follows easily.

Corollary 4.1.30 now lets us conclude the proof. $\square$

If X is a scheme where quasi-compact opens have finitely many irreducible components then in light of Lemma 4.2.2 we get a morphism

$$\mu^\eta : X^\eta = \underline{\mathrm{Spec}}_X(\mathscr{M}_\eta(X)) \to X \tag{4.2.8}$$

such that if $f : X \to Y$ is a morphism of schemes whose quasi-compact opens have finitely many irreducible components then we have a morphism $f^\eta : X^\eta \to Y^\eta$ fitting into a commutative diagram

$$\begin{array}{ccc} X^\eta & \xrightarrow{f^\eta} & Y^\eta \\ {\scriptstyle \mu^\eta(X)}\downarrow & & \downarrow{\scriptstyle \mu^\eta(Y)} \\ X & \xrightarrow{f} & Y. \end{array} \tag{4.2.9}$$

Proposition 4.2.3. *Let X be a scheme such that any quasi-compact open subset has finitely many irreducible components. Then the following statements hold true:*

1. *The morphism $\mu^\eta(X) : X^\eta \to X$ is integral.*
2. *The morphism $\mu^\eta(X)$ induces a bijection of underlying sets.*
3. *For any $x' \in X^\eta$ with $x = \mu^\eta(X)(x') \in X$ the map $\eta(k(x)) \to F(k(x))$ factors through the the map of residue fields $\overline{(\mu^\eta(X))}_{x'/x} : k(x) \to k(x')$. In particular $\mu^\eta(X)$ is a universal homeomorphism.*
4. *The map $\mathscr{N}_\eta(\mu^\eta(X)) : \mathscr{N}_\eta(X) \to (\mu^\eta(X))_*\mathscr{N}_\eta(X^\eta)$ is an isomorphism.*

Proof. Follows immediately from Lemma 4.2.2 and Proposition 4.1.10. $\square$

Corollary 4.2.4. *Let $f : X \to Y$ be a morphism of schemes where quasi-compact opens have finitely many irreducible components. Then:*

1. *f is integral if and only if the induced morphism f^η is integral.*
2. *f is surjective if and only if f^η is surjective.*

3. *f^η is an isomorphism if and only if f is affine and the morphism $\mathscr{M}_\eta(f)$: $\mathscr{M}_\eta(Y) \to f_*\mathscr{M}_\eta(X)$ is an isomorphism.*

   Moreover if the equivalent conditions of (3) are satisfied then we also have:

4. *The morphism f induces a bijection of underlying sets.*

5. *For any $x \in X$ with $y = f(x)$ the map $\eta(k(y)) : k(y) \to F(k(y))$ factors through the induced map of residue fields $\overline{f}_{x/y} : k(y) \to k(x)$.*

Proof. **For (1):** Suppose that f is integral. Then since $f \circ \mu^\eta(X)$ and the diagonal $\delta_{Y^\eta/Y}$[1] are both integral it follows that f^η is integral (one can also deduce this from the fact that $\mu^\eta(Y)$ is integral without considering the diagonal). Conversely if f^η is integral then $f \circ \mu^\eta(X)$ is also integral. From which it follows that if U is any affine open of Y then the induced map $U \times_Y X^\eta \to U \times_Y X$ is a surjective integral morphism from an affine scheme hence by [Stacks, Tag 05YU] the scheme $U \times_Y X$ is affine proving that f is an affine morphism. Now we can apply Corollary 4.1.11 to see that f is integral.

For (2): This follows immediately from part (2) of Proposition 4.2.3.

For (3): It is clear from the construction of f^η that if this is an isomorphism then $\mathscr{M}_\eta(f)$ is an isomorphism, furthermore in this case it follows from part (1) that f is integral hence affine. Conversely if f is affine with $\mathscr{M}_\eta(f)$ an isomorphism then from Corollary 4.1.11 we easily deduce that f is integral and induces a bijection of underlying sets. Thus f gives a homeomorphism of underlying topological spaces and so does f^η and since $(f^\eta)^\#$ is an isomorphism it follows that f^η is an isomorphism.

For (4): This is clear.

For (5): Follows from the affine case (Corollary 4.1.11). □

Corollary 4.2.5. *Let $f : X \to Y$ be morphism of schemes whose quasi-compact opens have finitely many irreducible components. Suppose in addition that f is integral and that $\mathscr{N}_\eta(f) : \mathscr{N}_\eta(Y) \to f_*\mathscr{N}_\eta(X)$ is an isomorphism. Then f^η is an isomorphism.*

Proof. By Part (3) of Corollary 4.2.4 it is enough to show that $\mathscr{M}_\eta(f)$ is an isomorphism. To this extent it is enough to show that $\mathscr{M}_\eta(f)(U)$ is an isomorphism whenever U is an affine open of Y. By Lemma 4.2.2 this now follows from Corollary 4.1.14. □

[1] Throughout this thesis we occasionally use the following basic fact without further mention: If $\mathcal{P}$ is a class of morphisms between schemes which is closed under composition and base change and $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_3$ are morphisms of schemes such that $f_2 \circ f_1$ is in $\mathcal{P}$ and the diagonal δ_{X_2/X_3} is in $\mathcal{P}$ then f_1 is necessarily also in $\mathcal{P}$ (see [Vak13, Thm.10.1.19]).

Corollary 4.2.6. *For a scheme X where quasi-compact opens have finitely many irreducible components the morphism $\mu(X^\eta) : (X^\eta)^\eta \to X^\eta$ is an isomorphism.*

Proof. Follows from part (4) of Proposition 4.2.3 and Corollary 4.2.5. □

Proposition 4.2.7. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Then for any integral morphism $f : Y \to X$ such that $\mathscr{N}_\eta(f) : \mathscr{N}_\eta(X) \to f_*\mathscr{N}_\eta(Y)$ is an isomorphism there exists a unique morphism $\tau(f) : X^\eta \to Y$ such that $\mu^\eta(X) = f \circ \tau(f)$.*

Proof. Existence follows immediately from Corollary 4.2.5. To prove the uniqueness of $\tau(f)$ it is enough to prove that if g is any other morphism satisfying $\mu^\eta(X) = f \circ g$ then $\tau(f)$ and g must agree over an affine open cover of X. This follows from Proposition 4.1.17. □

Lemma 4.2.8. *Let $f : X \to Y$ be a morphism of schemes whose quasi-compact opens have finitely many irreducible components. Then $f^\eta : X^\eta \to Y^\eta$ is the unique morphism that can fill the dotted arrow in the following commutative diagram*

$$\begin{tikzcd} X^\eta \arrow[r, dotted] \arrow[d, "\mu^\eta(X)"] & Y^\eta \arrow[d, "\mu^\eta(Y)"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{4.2.10}$$

Proof. Since X^η is reduced and $\mu^\eta(Y)$ radicial it follows immediately from Corollary 1.4.29. □

Lemma 4.2.9. *Let $p : X' \to X$ be a surjective morphism of schemes where quasi-compact opens have finitely many irreducible components. Then the induced morphism $p^\eta : (X')^\eta \to X^\eta$ is an epimorphism in the category of schemes.*

Proof. Follows because p^η is a surjective morphism to a reduced scheme. □

Proposition 4.2.10. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Suppose that $f : Y \to X^\eta$ is an integral morphism from a reduced scheme Y such that $\mathscr{N}_\eta(f)$ is an isomorphism. Then f is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X^\eta} \arrow[r] \arrow[d, "q_{X^\eta}"] & f_*\mathcal{O}_Y \arrow[d, "f_*q_Y"] \\ \mathscr{M}_\eta(X^\eta) \arrow[r, "\mathscr{M}_\eta(f)"] & f_*\mathscr{M}_\eta(Y). \end{tikzcd} \tag{4.2.11}$$

By Corollary 4.2.6 the morphism q_{X^η} is an isomorphism and by Corollary 4.2.5 and Corollary 4.2.4 it follows that $\mathscr{M}_\eta(f)$ is an isomorphism. Thus for any open subset U of X the map $f_*(q_Y)$ is surjective and since Y is reduced it is in fact an isomorphism. Now since f is integral hence closed and induces a bijection of underlying sets it follows that f gives a homeomorphism of underlying topological spaces. Thus q_Y is an isomorphism from which we conclude that f must be an isomorphism. □

Lemma 4.2.11. *Let $f : X \to Y$ be a morphism of schemes where every quasi-compact open subset has finitely many irreducible components. Then the induced morphism $f^\eta : X^\eta \to Y^\eta$ is universally closed if and only if f is.*

Proof. Suppose first that f is closed. Then since both $\mu^\eta(X)$ and $\mu^\eta(Y)$ are universal homeomorphisms it follows that $\mu^\eta(Y) \circ f^\eta$ is universally closed. Moreover since the diagonal morphism $\delta_{Y^\eta/Y}$ is also universally closed it follows that f^η must necessarily be universally closed.

For the converse statement suppose that f^η is universally closed and let $Z \to Y$ be any morphism of schemes. The commutative diagram from Lemma 4.2.8 induces the following commutative diagram:

$$\begin{tikzcd} Z \times_Y X^\eta \arrow[r] \arrow[d] & Z \times_Y Y^\eta \arrow[d] \\ Z \times_Y X \arrow[r] & Z \end{tikzcd}$$

where the vertical maps are universal homeomorphisms and the upper horizontal map is closed by assumption. We now easily conclude that the lower horizontal map must also be closed.

□

We summarise the main results in this section here:

Theorem 4.2.12. *Let X be a scheme whose quasi-compact open subsets have finitely many irreducible components. The following statements hold true:*

1. *The morphism $\mu^\eta(X) : X^\eta \to X$ is a universal homeomorphism such that for any point $x' \in X^\eta$ lying over $x \in X$ the field extension $\overline{\eta(k(x))} : k(x) \to F(k(x))$ factors through the induced map of residue fields $\overline{\mu^\eta(X)}_{x'/x} : k(x) \to k(x')$.*

2. *For any integral morphism $f : Y \to X$ such that $\mathscr{N}_\eta(f) : \mathscr{N}_\eta(X) \to f_*\mathscr{N}_\eta(Y)$ is an isomorphism there exists a unique morphism $\tau(f) : X^\eta \to Y$ such that $\mu^\eta(X) = f \circ \tau(f)$.*

3. *If $f : Y \to X^\eta$ is an integral morphism from a reduced scheme Y with $\mathscr{N}_\eta(f)$ an isomorphism then f is an isomorphism.*

4. *If $f : X \to Y$ is a morphism to a scheme Y whose quasi-compact open subsets have finitely many irreducible components then there exists a unique morphism $f^\eta : X^\eta \to Y^\eta$ making the following diagram commutative:*

$$\begin{tikzcd} X^\eta \arrow[r, dotted] \arrow[d, "\mu^\eta(X)"] & Y^\eta \arrow[d, "\mu^\eta(Y)"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{4.2.12}$$

In particular $\mu^{Id}X : X^{Id} \to X$ is the semi-normalization of X and $\mu^{(-)^{Perf}}X : X^{(-)^{Perf}} \to X$ is the absolute weak normalization of X.

Proof. 1 follows from Proposition 4.2.3, 2 is Proposition 4.2.7, 3 is Proposition 4.2.10 and finally the last part 4 follows from Lemma 4.2.8. □

Also integrality, surjectivity and universal closedness are preserved and reflected by the $(-)^\eta$ functor:

Proposition 4.2.13. *Let $f : X \to Y$ be a quasi-compact morphism of schemes whose quasi-compact opens have finitely many irreducible components. Then f is integral (resp. surjective, resp. universally closed) if and only if f^η is.*

Proof. The statement concerning integrality and surjectivity were both proved in Corollary 4.2.4. The final statement regarding universal closedness is exactly Lemma 4.2.11. □

The relative case

Construction 4.2.14. For a surjective affine morphism of schemes $g : Z \to X$ whose quasi-compact open subsets have finitely many irreducible components we let $\mathscr{M}_\eta(Z/X)$ be the pullback of $g_*q_Z : g_*\mathcal{O}_Z \to g_*\mathscr{M}_\eta(Z)$ along $\mathscr{M}_\eta(g) : \mathscr{M}_\eta(X) \to g_*\mathscr{M}_\eta(Z)$. Note that $\mathscr{M}_\eta(Z/X)$ considered as a sheaf of $\mathcal{O}_X$-modules is easily seen to be isomorphic to the kernel of the following map

$$\mathscr{M}_\eta(X) \times g_*\mathcal{O}_Z \overset{\mathscr{M}_\eta(g) - g_*q_Z}{\to} g_*\mathscr{M}_\eta(Z) \tag{4.2.13}$$

hence it follows from Lemma 4.2.2 that $\mathscr{M}_\eta(Z/X)$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.

We set $(Z/X)^\eta := \underline{\operatorname{Spec}}_X(\mathscr{M}_\eta(Z/X)) \overset{\mu^\eta(Z/X)}{\to} X$.

More generally for a surjective quasi-compact and quasi-separated morphism of schemes $g : Z \to X$ whose quasi-compact open subsets have finitely many irreducible components we set

$$\mathscr{M}_\eta(Z/X) := \mathscr{M}_\eta(\underline{\operatorname{Spec}}_X(g_*\mathcal{O}_Z)/X) \tag{4.2.14}$$

and $(Z/X)^\eta := \underline{\operatorname{Spec}}_X(\mathscr{M}_\eta(Z/X)) \overset{\mu^\eta(Z/X)}{\to} X$. Note that we have an obvious monomorphism $\mathscr{M}_\eta(Z/X) \to g_*\mathcal{O}_Z$ which thus induces an X-morphism $\underline{g} : Z \to (Z/X)^\eta$.

Proposition 4.2.15. *Let $g : Z \to X$ a surjective quasi-compact and quasi-separated morphism of schemes whose quasi-compact open subsets have finitely many irreducible components. Then the following statements hold true:*

1. *The morphism $\mu^\eta(Z/X) : (Z/X)^\eta \to X$ is integral.*

2. *The morphism $\mu^\eta(Z/X)$ induces a bijection of underlying sets.*

3. *For any $x' \in (Z/X)^\eta$ with $x = \mu^\eta(Z/X)(x') \in X$ the map $\underline{\eta(k(x))}$: $k(x) \to F(k(x))$ factors through the map of residue fields $\overline{\mu^\eta(Z/X)}_x$: $k(x) \to k(x')$.*

4. *The map $\mathscr{N}_\eta(\mu^\eta(Z/X)) : \mathscr{N}_\eta(X) \to \mu^\eta(Z/X)_* \mathscr{N}_\eta((Z/X)^\eta)$ is an isomorphism.*

5. *For any factorization of g of the form*

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[rr, bend left, "g"] & Y \arrow[r, "f"] & X \end{tikzcd} \tag{4.2.15}$$

where f is integral and the map $\mathscr{N}_\eta(f) : \mathscr{N}_\eta(X) \to f_* \mathscr{N}_\eta(Y)$ is an isomorphism there exists a unique map $\tau(f) : (Z/X)^\eta \to Y$ such that $\mu^\eta(Z/X) = f \circ \tau(f)$.

Proof. Parts (1) through (4) follow directly from Proposition 4.1.19. For the last part note that since the f is an affine morphism we can replace the morphism $g : Z \to X$ with $\underline{\mathrm{Spec}}_X(g_* \mathcal{O}_Z \to X)$ and hence assume that g is affine. From Corollary 4.2.5 it follows that the map $\mathscr{M}_\eta(f) : \mathscr{M}_\eta(X) \to f_* \mathscr{M}_\eta(Y)$ is an isomorphism. Thus we can consider the map

$$\mathscr{M}_\eta(f)^{-1} \circ f_*(q_Y) : f_* \mathcal{O}_Y \to \mathscr{M}_\eta(X) \tag{4.2.16}$$

which together with $f_*(f'^\#) : f_* \mathcal{O}_Y \to g_* \mathcal{O}_Z$ uniquely determine a morphism of $\mathcal{O}_X$-algebras $f_* \mathcal{O}_Y \to \mu^\eta(Z/X)_*(\mathcal{O}_{(Z/X)^\eta})$ and hence give us an X-morphism $\tau(f) : (Z/X)^\eta \to Y$. The uniqueness of this morphism follows easily from the affine case (Proposition 4.1.19). □

Corollary 4.2.16. *Let $g : Z \to X$ be a surjective quasi-compact and quasi-separated morphism of schemes whose quasi-compact open subsets have finitely many irreducible components. Suppose that g factors as $Z \xrightarrow{f'} Y \xrightarrow{f} X$ were f is integral and the map $\mathscr{N}_\eta(f)$ is an isomorphism. Then the schemes $(Z/X)^\eta$ and $(Z/Y)^\eta$ are canonically isomorphic.*

Proof. Follows easily from the universal property (Proposition 4.2.15 part Item 5). □

Lemma 4.2.17. *Let $p : Z \to X$ be a surjective quasi-compact and quasi-separated morphism of schemes whose quasi-compact opens have finitely many irreducible components. The following are equivalent:*

1. *The morphism $\mu^\eta(Z/X) : (Z/X)^\eta \to X$ is an isomorphism.*

2. *For every affine open U of X the induced morphism $U \times_X (Z/X)^\eta \to U$ is an isomorphism.*

3. *There exists an affine cover $\{U_i\}$ of X such that the induced morphism $U_i \times_X (Z/X)^\eta \to U_i$ is an isomorphism for every i.*

Proof. Clearly (1) implies (2) which again implies (3). Furthermore by Lemma 4.1.31 it follows from affine communication that (2) and (3) are in fact equivalent. Finally if (2) holds then it follows easily that $\mu^\eta(Z/X) : (Z/X)^\eta \to X$ is a homeomorphism with $\mu^\eta(Z/X)^\# : \mathcal{O}_X \to \mu^\eta(Z/X)_* \mathcal{O}_{(Z/X)^\eta}$ an isomorphism thus $\mu^\eta(Z/X)$ is an isomorphism. □

Faithfully flat descent for schemes

Lemma 4.2.18. *Let $f : Y \to X$ be a faithfully flat morphism locally of finite presentation. The following statements hold true:*

1. *If every quasi-compact open subset of X has finitely many irreducible components then so does every quasi-compact open subset of Y.*

2. *Any affine open subset U of X has an affine cover $U = \cup_{i \in I} \operatorname{Spec}(A_i)$ such that for each $i \in I$ there is some affine open subset $\operatorname{Spec}(B_i)$ of Y such that $f(\operatorname{Spec}(B_i)) = \operatorname{Spec}(A_i)$ or in other words the corresponding map of rings $A_i \to B_i$ is faithfully flat.*

Proof. **For (1):** Let V be any quasi-compact open subset of Y then since f is (universally) open ([Stacks, Tag 01UA]) it follows that $f(V)$ is open and we easily see that it is also quasi-compact. Furthermore since f is necessarily locally of finite type and generic points are mapped to generic points it follows then easily that any quasi-compact open contained in $f^{-1}(f(V))$ must necessarily have finitely many irreducible components, thus we conclude that V has finitely many irreducible components. **For (2):** We may reduce to the case where X is affine, say $X = \operatorname{Spec}(A)$. For any point $x \in X$ let $\operatorname{Spec}(B)$ be an affine open of Y such that the open subset $f(\operatorname{Spec}(B))$ contains the point x. We can then find some $g \in A$ such that $x \in D(g)$ and $D(g) \subset f(\operatorname{Spec}(B))$. This then easily implies that the induced map $\operatorname{Spec}(B_g) \to \operatorname{Spec}(A_g)$ is surjective. Since $x \in X$ was arbitrary we can conclude. □

Corollary 4.2.19 (Faithfully flat descent)**.** *Let $p : Z \to X$ be a surjective integral morphism of schemes whose quasi-compact opens have finitely many* *irreducible components. Suppose that $f : Y \to X$ is a faithfully flat morphism locally of finite presentation from a scheme Y. If $\mu^\eta(Y \times_X Z/Y)$ is an isomorphism then so is $\mu^\eta(Z/X)$.*

Proof. Suppose that $\mu^\eta(Y \times_X Z/Y)$ is an isomorphism. To check if $\mu^\eta(Z/X)$ is an isomorphism we may by Lemma 4.2.17 check this on an arbitrary affine cover hence by Lemma 4.2.18 2 we may assume that all schemes Y, X and Z are affine. The result follows now from Proposition 4.1.26. □

We denote the normalization of a scheme X by X^n.

Corollary 4.2.20. *Let X be a scheme whose quasi-compact opens have finitely many irreducible components and $f : Y \to X$ be a faithfully flat morphism locally of finite presentation. If the morphism*

$$\mu^\eta(Y^n/Y) : (Y^n/Y)^\eta \to Y \tag{4.2.17}$$

is an isomorphism then so is

$$\mu^\eta(X^n/X) : (X^n/X)^\eta \to X \tag{4.2.18}$$

Proof. By Lemma 4.2.17 this can be checked on an arbitrary cover of X and by Lemma 4.2.18 2 we reduce to the case where Y and X are affine. By Corollary 4.1.27 we are done. □

Relations with Manaresi's description

Lemma 4.2.21. *Suppose X_1, X_2 and X_3 are locally ringed spaces $p : X_1 \to X_2, f_1 : X_1 \to X_3$ morphisms of locally ringed spaces where the map p is surjective. If $f_2 : X_2 \to X_3$ is a morphism of ringed spaces such that $f_1 = f_2 \circ p$ then f_2 is necessarily a morphism of locally ringed spaces.*

Proof. This is easy. □

Lemma 4.2.22. *Suppose that X and Z are schemes whose quasi-compact open subsets have finitely many irreducible components. If $p : Z \to X$ is a morphism satisfying the following conditions:*

1. *p is quasi-compact and quasi-separated.*

2. *There exists a surjective quasi-compact universally closed morphism $p' : Z' \to X$ and an open cover $Z' = \cup_{i \in I} U_i$ such that for every $i \in I$ the restriction $p'|_{U_i}$ factors through p. In particular p is necessarily surjective.*

3. *For the fixed natural transformation $\eta : Id_{Fields} \to F$ we have that for any point $x \in X$ there exists some $z_x \in p^{-1}(\{x\})$ such that if $(k(z_x))_{pi} \subset k(z_x)$ denotes the purely inseparable closure of $k(x)$ in $k(z_x)$ then the induced map $F(k(x)) \to F((k(z_x))_{pi})$ is an isomorphism.*

Then

$$(Z \times_X Z)_{red} \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} Z \xrightarrow{\underline{p}} (Z/X)^\eta \tag{4.2.19}$$

is a co-equalizer diagram in the category of schemes.

Proof. To see that the maps $p_1 = \underline{p} \circ pr_1$ and $p_2 = \underline{p} \circ pr_2$ coincide let $z' \in (Z \times_X Z)_{red}$ be any point and note that since $\mu^\eta(Z/X) : (Z/X)^\eta \to X$ is a bijection of sets we must necessarily have $p_1(z') = p_2(z') \in (Z/X)^\eta$ and we denote this common point by x'. Furthermore from our standing assumption on the natural transformation η and Proposition 4.2.15 Item 3 it follows that if x is the image of x' in X then the induced map of residue fields $k(x) \to k(x')$ is purely inseparable hence it follows that p_1 and p_2 must agree on the level of residue fields as well thus $p_1 = p_2$.

Now suppose that $f : Z \to Y$ is any morphism such that $f \circ pr_1 = f \circ pr_2 : (Z \times_X Z)_{red} \to Y$. Note that this implies in particular that if $p(z_1) = p(z_2) \in X$ then $f(z_1) = f(z_2) \in Y$.

Thus given $x \in X$ we set $(f_X)_{top}(x) := f(z_x)$. This gives us a map of sets $(f_X)_{top} : |X| \to |Y|$. Hence on the level of maps of underlying sets we have $f = (f_X)_{top} \circ p$. Thus for any subset $Y' \subset Y$ we have $f^{-1}(Y') = p^{-1}((f_X)^{-1}_{top}(Y'))$. By assumption (2) it follows easily that the map of underlying topological spaces $|p| : |Z| \to |X|$ is a quotient map hence we conclude that the map of sets $(f_X)_{top}$ is continuous.

Pick now $x \in X$ and let $z_x \in Z$ be an element lying over x satisfying the condition given in (3), i.e. $F(k(x)) \to F(k(z_x)_{pi})$ is an isomorphism. By assumption and Lemma 1.4.23 we have that the image of $k(f(z_x)) \to k(z_x)$ is contained in $k(z_x)_{pi}$. If $z \in Z$ is any other element lying over x then $f(z) = f(z_x)$. Denote $f(z)$ by y. If we consider the diagram

$$\begin{tikzcd}
k(y) \arrow[r] \arrow[d] & k(z_x)_{pi} \arrow[r] \arrow[d] & k(z_x) \arrow[d] \\
k(z) \arrow[r] & (k(z) \otimes_{k(x)} k(z_x)_{pi})_{red} \arrow[r, hook] & (k(z_x) \otimes_{k(x)} k(z))_{red}
\end{tikzcd} \tag{4.2.20}$$

we have that the outer rectangle commutes by assumption from which we easily deduce that the leftmost square is commutative. Then since $(k(z) \otimes_{k(x)} k(z_x)_{pi})_{red}$ is a field it follows that the induced map $F(k(y)) \to F(k(z))$ coincides with the compositions

$$F(k(y)) \to F(k(z_x)_{pi}) \overset{\cong}{\to} F(k(x)) \overset{F(\overline{p_x})}{\to} F(k(z)). \tag{4.2.21}$$

From this one easily concludes that if $z'_x \in Z$ is any other point lying over x satisfying the condition given in ((3)), then we have an equality

$$(F(k(y)) \to F(k(z_x)_{pi}) \cong F(k(x))) = (F(k(y)) \to F(k(z'_x)_{pi}) \cong F(k(x))). \tag{4.2.22}$$

We denote this common map by $\gamma_x : F(k((f_X)_{top}(x))) \to F(k(x))$.

Now for V an open subset of Y we let $\gamma(V) : \mathscr{N}_\eta(Y)(V) \to ((f_X)_{top})_*\mathscr{N}_\eta(X)(V)$ be the unique map such that for any $x \in ((f_X)_{top})^{-1}(V)$ with $y = (f_X)_{top}(x) \in V$ the diagram

$$\begin{tikzcd} \mathscr{N}_\eta(Y)(V) \arrow[r] \arrow[d] & \mathscr{N}_\eta(X)((f_X)^{-1}_{top}(V)) \arrow[d] \\ F(k(y)) \arrow[r, "\gamma_x"] & F(k(x)) \end{tikzcd} \tag{4.2.23}$$

commutes. This gives us a map of sheaves of rings $\gamma : \mathscr{N}_\eta(Y) \to ((f_X)_{top})_*\mathscr{N}_\eta(X)$. Note that

$$\mathscr{N}_\eta(f) = \mathscr{N}_\eta(p) \circ \gamma. \tag{4.2.24}$$

We will now show that the restriction of $\mathscr{N}_\eta(f)$ to $\mathscr{M}_\eta(Y)$ factors through $f_*\mathscr{M}_\eta(X)$. To this extent suppose that $g : \mathrm{Spec}(R) \to X$ is a morphism with R a valuation ring with $\eta(R_{(0)})$ an isomorphism. We need to show that for any open V of Y the image of $[((f_X)_{top})_*(N_f(g))] \circ \gamma(V)|_{\mathscr{M}_\eta(Y)(V)}$ is contained in the image of $((f_X)_{top})_* g_*(t_{\mathrm{Spec}(R)})(V)$. By assumption 2 and Lemma Lemma 1.3.12 it follows easily that we can find a valuation ring R' together with maps $g' : \mathrm{Spec}(R') \to Z$ and $h : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ such that $p \circ g' = g \circ h$ with h surjective. We claim that the map $f \circ g'$ can be factored through h. Indeed if z is the image of the generic point of $\mathrm{Spec}(R')$ in Z and $y = f(z) \in Y$, $x = p(z) \in X$ then we have a diagram where each square commutes

$$\begin{tikzcd} k(y) \arrow[rr] \arrow[ddd] & & R'_{(0)} \arrow[r, "id_{R'_{(0)}}"] & R'_{(0)} \arrow[ddd] \\ & & R_{(0)} \arrow[u] & \\ & & & \\ F(k(y)) \arrow[r] & F(k(x)) \arrow[r] & F(R_{(0)}) \arrow[r] \arrow[uu, "\cong"] & F(R'_{(0)}). \end{tikzcd} \tag{4.2.25}$$

Thus by picking an affine open of Y say $\mathrm{Spec}(A)$ containing the image of $f \circ g'$ we see that the image of the induced morphism of rings $A \to R'$ must be contained in $R' \cap R_{(0)} = R$ from which we deduce that there exists a morphism $g'' : \mathrm{Spec}(R) \to Y$ such that $f \circ g' = g'' \circ h$. Since $\mathscr{N}_\eta(f) = ((f_X)_{top})_*\mathscr{N}_\eta(p) \circ \gamma$ and since ${g''}_*\mathscr{N}_\eta(h)$ is a monomorphism we conclude the equality

$$((f_X)_{top})_*(\mathscr{N}_\eta(g)) \circ \gamma = \mathscr{N}_\eta(g''). \tag{4.2.26}$$

Thus for any open V of Y the image of $[((f_X)_{top})_*(\mathscr{N}_\eta(g))] \circ \gamma(V)|_{\mathscr{M}_\eta(Y)(V)}$ is contained in the image of $((f_X)_{top})_*(t_{\mathrm{Spec}(R)})(V)$ which now immediately implies that the restriction of $N_f(f)$ to $\mathscr{M}_\eta(Y)$ factors through $f_*\mathscr{M}_\eta(X)$. Thus we have a map $\tau : \mathscr{M}_\eta(Y) \to (f_X)_{top_*}\mathscr{M}_\eta(X)$ such that

$$\mathscr{M}_\eta(f) = (f_X)_{top_*}\mathscr{M}_\eta(p) \circ \tau. \tag{4.2.27}$$

Thus by the definition of $\mathscr{M}_\eta(Z/X)$ we get induced a morphism

$$(f')^\# : \mathcal{O}_Y \to (f_X)_{top_*}\mathscr{M}_\eta(Z/X) \cong (f_X)_{top_*}\mu^\eta(Z/X)_*\mathcal{O}_{(Z/X)^\eta}, \tag{4.2.28}$$

hence we get a map of ringed spaces

$$f' := (f_X \circ \mu^\eta(Z/X), f'^\#) : (Z/X)^\eta \to Y \tag{4.2.29}$$

such that $f = f' \circ \underline{p}$. It follows immediately from Lemma 4.2.21 that f' is a morphism of schemes. Uniqueness of f' follows because $\underline{p}$ is an epimorphism in the category of schemes (being surjective and giving an injection on the level of sheaves). □

Remark 4.2.23. The condition (2) of Lemma 4.2.22 can be replaced with requiring the morphism p to subtrusive; since such morphisms yield a quotient map of the underlying topological spaces and they have a valuative criterion (Remark 1.3.13) the same proof will work. Moreover we emphasize that [Ryd10, Thm. 7.4] is essentially a non-Noetherian generalization of Lemma 4.2.22.

In the case of rings with finitely many minimal prime ideals we now get another proof of Manaresi's characterization of the weak normalization [Man80, Thm. I.6].

Corollary 4.2.24 (Manaresi)**.** *Let $A \subset B$ be an integral extension of rings both of which have finitely many minimal prime ideals. Let $*_B(A)$ denote the weak normalization of A in B. Then the following diagram is an equalizer:*

$$*_B(A) \longrightarrow B \underset{(\otimes 1)_{red}}{\overset{(1\otimes)_{red}}{\rightrightarrows}} (B \otimes_A B)_{red} \tag{4.2.30}$$

Proof. Letting F be the perfect closure $(-)^{Perf}$ the result follows immediately from Lemma 4.2.22. □

Corollary 4.2.25. *Suppose that $p : Z \to X$ is a surjective quasi-compact, quasi-separated and universally closed of schemes whose quasi-compact opens have finitely many irreducible components. Then if every point $x \in X$ has a point $z_x \in Z$ lying over x such that the induced extension $k(x) \to (k(z_x))_{pi}$ is an isomorphism then we have the equality*

$$(Z/X)^{Id} = (Z/X)^{(-)^{Perf}}. \tag{4.2.31}$$

In particular if $A \to B$ is an integral extension of rings where every point $p \in \operatorname{Spec}(A)$ has a point $q \in \operatorname{Spec}(B)$ lying over p such that the induced extension $k(p) \to (k(q))_{pi}$ is trivial then the semi and weak normalizations of A in B coincide.

Proof. Follows immediately from Lemma 4.2.22. □

Remark 4.2.26. Corollary 4.2.25 can also be deduced from the universal properties.

Limit description

In [Ryd10] it is explained that the absolute weak normalization of a scheme X is the limit over all universal homeomorphisms of finite presentation over X. We will now give a similar description of X^η.

Lemma 4.2.27. *Let X be a Noetherian scheme. The morphism*

$$\mu^\eta(X) : X^\eta \to X \tag{4.2.32}$$

is the directed limit of all finite morphisms $f_\lambda : X_\lambda \to X$ such that

$$\mathscr{N}_\eta(f_\lambda) : \mathscr{N}_\eta(X) \to f_{\lambda *}\mathscr{N}_\eta(X_\lambda) \tag{4.2.33}$$

is an isomorphism.

Proof. By [Stacks, Tag 0817] the quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{M}_\eta(X)$ is the direct colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras. If Λ is any such finite quasi-coherent $\mathcal{O}_X$-subalgebra, then letting $f_\lambda : \underline{\mathrm{Spec}}_X(\Lambda) \to X$ be the induced map it is clear that $\mu^\eta(X)$ factors through f_λ and that $X^\eta \to \underline{\mathrm{Spec}}_X(\Lambda)$ is surjective. Thus $\mathscr{N}_\eta(f_\lambda)$ is an isomorphism.

To complete the proof it is thus enough to show that if $f : Y \to X$ is any finite morphism with $\mathscr{N}_\eta(f)$ an isomorphism then there exists some finite quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathscr{M}_\eta(X)$ which we denote Λ' such that $\underline{\mathrm{Spec}}_X(\Lambda')$ factors through f. To this extent note that by Corollary 4.2.5 it follows that $\Lambda' := f_*(\mathcal{O}_{Y red})$ is a finite quasi coherent sub-sheaf of $\mathscr{M}_\eta(X)$ satisfying the desired property. □

4.3 Applications of the η-construction to representable sheaves

From this point on until the end of the thesis every scheme is assumed to be separated. Moreover in this section S will always denote a Noetherian scheme unless we state otherwise.

Describing representable sheaves in non-subcanonical topologies

In this last part of the chapter we will for a suitable topology t describe the sheafification of a representable presheaf in terms of the pointwise η-construction. The main ideas behind the proof are not so different from those of [Ryd10, Theorem 8.16] in the sense that we both essentially reduce the argument to a certain diagram being a co-equalizer and then apply a limit argument to conclude the proof.

We will need the following lemma:

Lemma 4.3.1. *The following assertions hold true:*

1. *For a surjective quasi-compact and quasi-separated morphism $p : Z \to X$ of schemes where quasi-compact opens have finitely many irreducible components we have an isomorphism $(Z^\eta/X^\eta)^\eta \cong X^\eta$.*

2. *Suppose that $T_1 \to T, T_2 \to T$ are two finite type morphisms to a locally Noetherian scheme T Then the canonical morphism*

$$(T_1 \underset{T}{\times} T_2)^\eta \to ({T_1}^\eta \underset{T^\eta}{\times} {T_2}^\eta)_{red} \tag{4.3.1}$$

is an epimorphism in the category of schemes

Proof. The first assertion follows immediately from Proposition 4.2.10. For the second assertion it is enough to show that the map

$$(T_1 \underset{T}{\times} T_2)^\eta \to ({T_1}^\eta \underset{T^\eta}{\times} {T_2}^\eta)_{red} \tag{4.3.2}$$

is surjective. To this extent note first that by the universal property of $T_1 \times_T T_2$ we get an induced map

$$\pi : {T_1}^\eta \underset{T^\eta}{\times} {T_2}^\eta \to T_1 \underset{T}{\times} T_2 \tag{4.3.3}$$

and the map

$$\mu^\eta(T_1 \underset{T}{\times} T_2) : (T_1 \underset{T}{\times} T_2)^\eta \to (T_1 \underset{T}{\times} T_2) \tag{4.3.4}$$

factors as

$$(T_1 \underset{T}{\times} T_2)^\eta \to {T_1}^\eta \underset{T^\eta}{\times} {T_2}^\eta \xrightarrow{\pi} T_1 \underset{T}{\times} T_2. \tag{4.3.5}$$

Hence it is enough to show that π gives a bijection of underlying sets. Surjectivity is easy. For injectivity suppose that

$$\gamma_1, \gamma_2 : \mathrm{Spec}(E) \to {T_1}^\eta \underset{T^\eta}{\times} {T_2}^\eta \tag{4.3.6}$$

are such that $\pi \circ \gamma_1 = \pi \circ \gamma_2$. Then by universal injectivity of ${T_i}^\eta \to T_i$ for $i = 1, 2$ it follows easily that we must have $\gamma_1 = \gamma_2$ which completes the proof. $\square$

Lemma 4.3.2. *Let t be a topology on the category of Noetherian schemes over S satisfying the following properties*

1. *If $\{T_i \overset{p_i}{\to} T\}_{i\in I}$ is a t-covering of the Noetherian scheme T then so is the induced morphism*
$$(p_i)_{i\in I} : \coprod_{i\in I} T_i \to T \tag{4.3.7}$$
2. *For every t-covering $p : W \to T$ the induced morphism $p^\eta : W^\eta \to T^\eta$ satisfies the conditions of Lemma 4.2.22.*

Then for any scheme X over S the presheaf on the category of Noetherian schemes over S
$$\mathrm{Hom}_S((-)^\eta, X) \tag{4.3.8}$$
is a t-sheaf.

Proof. Suppose we have a t-covering $\{T_i \overset{p_i}{\to} T\}_{i\in I}$ and morphisms $f_i : T_i{}^\eta \to X$ such that for every pair $i, j \in I$ letting pr_1, pr_2 denote the projections from $T_i \times_T T_j$ on the first and second factor respectively we have
$$f_i \circ pr_1{}^\eta = f_j \circ pr_2{}^\eta \tag{4.3.9}$$
Set $W := \coprod_{i\in I} T_i$ and note that we get induced S-morphisms $p = (p_i)_{i\in I} : \coprod T_i \to T$ and $f : W \to X$ such that $f|_{T_i} = f_i$ for all $i \in I$.

Since $(-)^\eta$ commutes with coproducts it follows easily that if q_1, q_2 denote the projections from $W \times_T W$ onto the first and second factor respectively then we must have
$$f \circ q_1{}^\eta = f \circ q_2{}^\eta. \tag{4.3.10}$$
Thus if p_1, p_2 denote the projection from $(W^\eta \times_{T^\eta} W^\eta)_{\mathrm{red}}$ onto the first and second factors respectively it follows from Lemma 4.3.1 that we must have
$$f \circ p_1 = f \circ p_2. \tag{4.3.11}$$
Since $p^\eta : W^\eta \to T^\eta$ satisfies the conditions of Lemma 4.2.22 by assumption, we get induced a unique morphism $f_T : (W^\eta/T^\eta)^\eta = T^\eta \to X$ such that $f = f_T \circ p$ hence f_T is also the unique morphism satisfying $f_i = f_T \circ p_i$ for every $i \in I$. Since p^η is an epimorphism in the category of schemes (being a surjective morphism to a reduced scheme) it follows that f_T is a morphism of S-schemes. Thus proving that $\mathrm{Hom}_S((-)^\eta, X)$ is a sheaf in the t-topology. □

Lemma 4.3.3. *Let t_1 be a finer topology than t_2 on a category $\mathcal{C}$. Suppose that $\mathcal{F} \to \mathcal{F}'$ is a sheafification with respect to t_2 and that $\mathcal{F}'$ is a sheaf with respect to t_1. Then $\mathcal{F} \to \mathcal{F}'$ is a sheafification with respect to t_1.*

Proof. This is easy. □

If t is a (Grothendieck) topology on the category of schemes over S (Sch/S) we shall denote by $t|_{Noeth}$ its restriction to the full subcategory of Noetherian schemes over S. This means that the coverings of $t|_{Noeth}$ are exactly those coverings of t which only involve Noetherian S-schemes. Further we let t_η be the topology on Sch/S where the coverings are (either empty) or the singletons of the form $\{p : U' \to U\}$ where p is a finite morphism with $\mathscr{N}_\eta(p)$ an isomorphism.

Proposition 4.3.4. *Let t be a topology on the category of schemes over S such that $t|_{Noeth}$ satisfies the conditions of Lemma 4.3.2. If the topology t is finer than t_η for some given $\eta : Id_{Fields} \to F$ then if X is a scheme locally of finite type over S the canonical morphism*

$$\begin{aligned}\Phi_{X/S} : h_{X/S} = \mathrm{Hom}_S(-, X) &\to \mathrm{Hom}_S((-)^\eta, X)\\ (f : T \to X) &\mapsto (f \circ \mu^\eta(T)).\end{aligned}$$

of presheaves on the category of Noetherian schemes over S is a $t'|_{Noeth}$-sheafification for any topology t' (on Sch/S) finer than t_η and coarser than t.

Proof. By Lemma 4.3.2 the presheaf

$$\mathrm{Hom}_S((-)^\eta, X) \tag{4.3.12}$$

is a $t|_{Noeth}$-sheaf. Hence by Lemma 4.3.3 it is enough to show that $\Phi_{X/S}$ is a $t_\eta|_{Noeth}$ sheafification. To this extent it is enough to show that $\Phi_{X/S}$ is both a local monomorphism and a local epimorphism. To show the former suppose that T is a Noetherian scheme over S and we have S-morphisms $f, g \in h_{X/S}(T)$ such that $\Phi_{X/S}(T)(f) = \Phi_{X/S}(T)(g)$. Then since $\mu^\eta(T)_{red} \to T_{red}$ is an epimorphism in the category of schemes we have that $f|_{T_{red}} = g|_{T_{red}}$ or in otherwords f and g pullback to the same S-morphism under $T_{red} \to T$ which is a $t_\eta|_{Noeth}$-covering of T.

To see that $\Phi_{X/S}$ is a local epimorphism let $f' \in \mathrm{Hom}_S(T^\eta, X)$ be an element of $\mathrm{Hom}_S((-)^\eta, X)(T)$. Then using Lemma 4.2.27 we can write T^η as a limit $T^\eta = \lim T_\lambda$ where the $p_\lambda : T_\lambda \to T$ are finite morphisms with $\mathscr{N}_\eta(p_\lambda)$ isomorphisms. Now since X is locally of finite presentation over S we have by Proposition 1.8.8 some S-morphism $f_\lambda : T_\lambda \to X$ such that f' coincides with the composition

$$T^\eta \to T_\lambda \xrightarrow{f_\lambda} X. \tag{4.3.13}$$

Furthermore by Proposition 4.2.7 and Lemma 4.2.8 it follows that

$$f_\lambda \circ \mu^\eta(T_\lambda) = f' \circ {p_\lambda}^\eta \tag{4.3.14}$$

which completes the proof. □

Notation 4.3.5. For a scheme X we shall from now on use the less clunky notation X^{sn} in place of X^{Id} (and for a morphism $f : X \to Y$ we use $f^{sn} : X^{sn} \to Y^{sn}$ in place of f^{Id}) . Similarly we will use X^{awn} in place of $X^{(-)^{Perf}}$ and $f^{awn} : X^{awn} \to Y^{awn}$ in stead of $f^{(-)^{Perf}} : X^{(-)^{Perf}} \to Y^{(-)^{Perf}}$.

Lemma 4.3.6. *Let $p : Z \to X$ be a sd-h-covering (resp. an h-covering) of Noetherian schemes . Then p^{sn} (resp. p^{awn}) satisfies the three conditions given in Lemma 4.2.22 for $\eta = Id$ (resp. $\eta = (-)^{Perf}$).*

Proof. The first condition (1) is checked straightforwardly (recall that we assume all schemes to be separated in this section). For the second condition note that by Theorem 3.1.16 we have a proper surjective morphism $p' : Z' \to X$ and a Zariski covering $Z' = \cup_{i \in I} U_i$ such that $p'|_{U_i}$ factors through p. By Lemma 4.2.11 the morphism $(p')^{sn}$ (resp. $(p')^{awn}$) is surjective and universally closed and since $(-)^{\eta}$ takes open embeddings to open embeddings we conclude that the second condition is also satisfied. Finally the last condition (3) is straightforwardly satisfied. $\square$

Definition 4.3.7. We define the *uh* (resp. *cd* − *uh*-topology) on Sch/S to be the topology where coverings of a scheme U are of the form $\{p : U' \to U\}$ where p is a finite universal homeomorphism (resp. a finite universal homeomorphism with trivial residue field extensions).

Remark 4.3.8. Note that *uh* (resp. *cd* − *uh*) is exactly $t_{(-)^{Perf}}$ (resp. t_{Id}) in the text preceding Proposition 4.3.4.

From this point on we restrict the h, sd-h, *uh* and *cd* − *uh* topologies to the category of Noetherian schemes over S and for ease of notation we shall still denote them by h, sd-h etc. (in place of $h|_{Noeth}$ etc.).

Theorem 4.3.9. *Let X be a scheme locally of finite type over S. Then the following statements hold true:*

1. *For any topology t on the category of Noetherian S-schemes finer than the cduh topology and coarser than the sd-h-topology the natural transformation*

$$\begin{aligned} h_{X/S} &\to \mathrm{Hom}_S((-)^{sn}, X) \\ (f : T \to X) &\mapsto f \circ \mu^{Id}(T). \end{aligned}$$

   of presheaves on the category of Noetherian schemes is a t-sheafification of $h_{X/S}$. Moreover for any Noetherian scheme T we have $\mathrm{Hom}_S(T^{sn}, X) = \mathrm{colim}_{T_\lambda} \mathrm{Hom}_S(T_\lambda, X)$ where the colimit is over all finite universal homeomorphisms with trivial residue field extensions of T.

2. *For any topology t on the category of Noetherian S-schemes finer than the uh topology and coarser than the h-topology the natural transformation*

$$\begin{aligned} h_{X/S} &\to \mathrm{Hom}_S((-)^{awn}, X) \\ (f : T \to X) &\mapsto f \circ \mu^{(-)^{Perf}}(T). \end{aligned}$$

of presheaves on the category of Noetherian schemes is a t-sheafification of $h_{X/S}$. Moreover for any Noetherian scheme T we have $\mathrm{Hom}_S(T^{awn}, X) = \mathrm{colim}_{T_\lambda} \mathrm{Hom}_S(T_\lambda, X)$ *where the colimit is over all finite universal homeomorphisms of T.*

Proof. We prove (1) and (2) simultanously. From Lemma 4.3.6 it follows that the $sd - h$ topology (resp. the h-topology) satisfies the conditions of Lemma 4.3.2 when $\eta = Id$ (resp. $\eta = (-)^{Perf}$). By Proposition 4.3.4 and Lemma 4.2.27 we are done. □

Remark 4.3.10.

1. Item 1 of this Theorem extends most of the statement of ([HK18, Prop. 6.14]) (the isomorphism to $h_{X/S_{val}}$ being the statement not considered by us).

2. Item 2 is a special case of [Ryd10, Thm. (8.16)], indeed in loc.cit. X may be taken to be an algebraic space locally of finite presentation over S, and the Noetherian conditon is also relaxed a little there.

More on representable sheaves in the h-topologies

Theorem 4.3.9 can be applied to give simple proofs of many of the results of Section 3.2 of [Voe96] and we can also deduce analogous results for several related topologies.

For a Grothendieck topology t we let $\mathrm{L}_t(X/S)$ denote the t-sheafification of the representable functor $h_{X/S}$. The following Corollary is a generalization of [Voe96, Prop.3.2.5].

Corollary 4.3.11. *Let $f : X \to Y$ be a morphism of finite type of schemes locally of finite type over S and t_1 any topology that is finer than the uh topology and coarser than the h-topology and t_2 any topology finer than the cduh topology and coarser than the sd-h topology. Then the following assertions hold true:*

1. *The morphism $\mathrm{L}_{t_1}(f)$ (resp. $\mathrm{L}_{t_2}(f)$) is a monomorphism in the category of t_1 (resp. t_2) sheaves if and only if f is radicial.*

2. *The morphism $\mathrm{L}_{t_1}(f)$ (resp. $\mathrm{L}_{t_2}(f)$) is an epimorphism of t_1 (resp. t_2) sheaves if and only if f can be refined by a t_1 (resp. t_2)-covering.*

3. *The morphism $\mathrm{L}_{t_1}(f)$ (resp. $\mathrm{L}_{t_2}(f)$) is an isomorphism if and only if f is a uh covering (resp. $cd - uh$ covering).*

Proof. **For (1):** Suppose first that f is radicial. Then if T is any Noetherian S-scheme and α, β are any two morphisms with source T^{awn} (resp. T^{sn}) and target X such that $f \circ \alpha = f \circ \beta$ then from Corollary 1.4.29 we get $\alpha = \beta$. Conversely suppose that f is not radicial, then by Lemma 1.4.28 we can find some field K and two morphisms $t_1, t_2 : \mathrm{Spec}(K) \to X$ such that $f \circ t_1 = f \circ t_2$ but $t_1 \neq t_2$. Moreover it is clear that this field can be taken to be perfect hence $\mathrm{L}_{t_1}(f)$ (resp. $L_{t_2}(f)$) is not a monomorphism.

For (2): This follows easily from the following general fact: If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on a category with a Grothendieck topology then the associated map of sheaves is an epimorphism if and only if ϕ is a local epimorphism with respect to this topology.

For (3): If f is a *uh* covering (resp. *cduh*) covering then it follows immediately from (1) and (2) that $\mathrm{L}_{t_1}(f)$ (resp. $L_{t_2}(f)$) is an isomorphism.

Conversely if $\mathrm{L}_{t_1}(f)$ (resp. $L_{t_2}(f)$) is an isomorphism. Then it follows from (1) that f is radicial and from (2) that f can be refined by a t_1 (resp. t_2) covering. Recalling that both the t_1 and t_2 topologies are coarser than the h-topology we see that since f is necessarily a radicial universal topological epimorphism it easily follows that f is a universal homeomorphism. It remains to show that a universal homeomorphism which is a covering in the sd-h-topology is necessarily a $cd - uh$ covering, but this is easy. $\square$

Corollary 4.3.12. *Let X and Y be schemes locally of finite type over S and suppose t is some topology where we have an isomorphism of t-sheaves*

$$\mathrm{L}_t(X/S) \overset{\cong}{\to} \mathrm{L}_t(Y/S). \tag{4.3.15}$$

If the topology t is finer than the uh (resp. $cd - uh$) topology and coarser than the h (resp. sd-h) topology then there exists an S-scheme Z and finite universal homeomorphisms (resp. finite universal homeomorphisms with trivial residue field extensions) of S-schemes $f_1 : Z \to X$ and $f_2 : Z \to Y$.

Proof. By Theorem 4.3.9 and Theorem 4.2.12 Part (4) it follows from Yoneda Lemma that we get induced an isomorphism $f' : X^{awn} \to Y^{awn}$ (resp. $f : X^{sn} \to Y^{sn}$) which canonically gives us a morphism of S-schemes f with source X^{awn} (resp. X^{sn}) and target Y. Furthermore by Lemma 4.2.27 and Proposition 1.8.8 there exists some finite universal homeomorphism $f_1 : Z \to X$ (resp. finite universal homeomorphism with trivial residue field extensions) and a morphism $f_2 : Z \to Y$ such that f factors through f_2. One verifies straightforwardly that f_2 is a universal homeomorphism (resp. universal homeomorphism with trivial residue field extensions) and since both Z and Y are locally of finite type over S it follows that f_2 must necessarily be finite. $\square$

The following Lemma is more or less [Voe96, Lemma.3.1.7].

Lemma 4.3.13. *Let Y be a Noetherian integral geometrically unibranch Nagata scheme. Let L be a finite purely inseparable field extension of the field of functions of Y denoted by K. Then if $f : X \to Y$ is the normalization of Y in L the morphism f is a universal homeomorphism.*

Proof. Note that $f : X \to Y$ is a finite surjective morphism (this is where we are using the Nagata hypothesis) between integral schemes. It is necessarily equidimensional of dimension zero and by Proposition 1.1.20 it is in fact universally equidimensional of dimension zero and therefore also necessarily universally open. By Lemma 1.4.30 and Lemma 1.4.28 it is enough to show that the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is surjective. Note that since f is universally open it follows that every irreducible component of $X \times_Y X$ surjects onto X under either of the two projections. From this we see that to prove that $\Delta_{X/Y}$ is surjective it is enough to show that the kernel of the multiplication map

$$L \otimes_K L \to L \tag{4.3.16}$$

is contained in the nilradical of $L \otimes_K L$. But since the kernel is generated by elements of the form $z \otimes_K 1 - 1 \otimes_K z$ with $z \in L$, and L/K is purely inseparable this is easy. □

Corollary 4.3.14 (See also [Voe96, Prop.3.2.11])**.** *Let Y/S be a Noetherian integral geometrically unibranch Nagata scheme over S. Let t be any topology finer than the uh-topology and coarser than the h-topology. Then we have*

$$\mathrm{L}_h(Y/S)(T) = \mathrm{colim}_L \operatorname{Hom}_S(T_L, Y)$$

where the colimit runs over all finite purely inseparable field extensions of the field of functions of T and T_L denotes the normalization of T in L.

Proof. Just note that if $T' \to T$ is a finite purely inseparable universal homeomorphism then if L is the function field of T', it follows from [Stacks, Tag 035I] that the normalization of T in L denoted $f : T_L \to T$ must factor through T'. By Lemma 4.3.13 and Theorem 4.3.9 we conclude the proof. □

Finally the following is a generalization of [Voe96, Proposition 3.2.12].

Corollary 4.3.15. *Let X be a scheme locally of finite type over S and let t be any topology which is either finer than the uh topology and coarser than the h-topology or finer than the $cd - uh$ topology and coarser than the sd-h-topology. Then the morphism*

$$h_{X/S} \to \mathrm{L}_t(X/S) \tag{4.3.17}$$

of presheaves on the category of Noetherian schemes is an isomorphism if and only if X is étale over S.

Proof. Suppose first that X is étale over S. Then since surjective morphisms of reduced schemes are necessarily epimorphisms it follows easily from the functorial characterisation of étale morphisms ([Stacks, Tag 025K]) that the map

$$h_{X/S} \to \mathrm{L}_t(X/S) \tag{4.3.18}$$

of presheaves is a monomorphism. To show that it is an epimorphism as well Theorem 4.3.9 tells us that it suffices to show that if $T' \to T$ is a finite universal homeomorphism of S-schemes with T' reduced and we are given an S-morphism $f : T' \to X$ then it must necessarily factor through T. To this extent we first have from [GD67, Corollaire (17.9.3)] that the graph $\Gamma_f : T' \to T' \times_S X$ is both an open and closed embedding, and moreover since $T' \times_S X \to T \times_S X$ is a universal homeomorphism it follows that the set theoretical image of Γ_f in $T \times_S X$ is an open and closed subset of $T \times_S X$ which we denote by Z. Note that the restriction of the projection $T \times_S X \to T$ to Z (considered as an open subscheme of $T \times_S X$) is necessarily surjective and universally injective, thus by loc.cit again we know that the map $T \times_S X \to T$ has a section $g : T \to T \times_S X$ which is necessarily a closed and open embedding with $g(T) = Z$. Furthermore by [GD67, Theorem (17.9.1)] we have that the map $Z \to T$ is necessarily an isomorphism hence f does indeed factor through T.

Conversely suppose that $h_{X/S} \to \mathrm{L}_t(X/S)$ is an isomorphism. To show that f is étale it is by [GD67, Proposition (17.14.2)] enough to check the functorial characterisation of étale morphisms with the additional hypothesis that the affine S-scheme being Noetherian. Thus suppose $Y = \mathrm{Spec}(A)$ is a Noetherian S-scheme and I a square-zero ideal and let $i : Y_0 := \mathrm{Spec}(A/I) \to Y$ be the closed embedding. Consider now the following commutative diagram

$$\begin{tikzcd} \mathrm{L}_t(X/S)(Y) \arrow[r, "i^*"] & \mathrm{L}_t(X/S)(Y_0) \\ h_{X/S}(Y) \arrow[u] \arrow[r, "i^*"] & h_{X/S}(Y_0) \arrow[u] \end{tikzcd} \tag{4.3.19}$$

Since $i^* : \mathrm{L}_t(X/S)(Y) \to \mathrm{L}_t(X/S)(Y_0)$ is an isomorphism by Corollary 4.3.11 and the vertical arrows are isomorphisms by assumption it follows that $h_{X/S}(Y) \to h_{X/S}(Y_0)$ is a bijection thus proving that X is étale over S. $\square$

Chapter 5

Representable monoids in the h-topology

The purpose of this chapter is to prove a theorem which will play a key role in the proof of our main Theorems. Indeed we want to compare relative cycles with morphisms to a commutative monoid in the category of schemes. The theorem to be proven in this chapter then allows us to transform this problem to working with h-sheaves instead. This is useful because we then have more tools to produce maps between our sheaves.

The precise result that we are going to prove is the following Theorem:

Theorem 5.0.1. *Let S be a Noetherian scheme and M/S be a commutative monoid object in the category of schemes over S and t be any Grothendieck topology finer than the uh topology and coarser than the h topology. Suppose further that the morphism $M \to S$ is flat, locally of finite type and* AF *(Definition 1.5.26). Then after restricting the presheaves $h_{M/S}$ and its t-sheafification $\mathrm{L}_t(M/S)$ to the category of semi-normal Noetherian S-schemes the natural map*

$$h_{M/S} \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{L}_t(M/S) \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{5.0.1}$$

becomes an isomorphism.

In light of Theorem 4.3.9 Item 2 we can informally say that Theorem 5.0.1 provides a bridge between the absolute weak normalization and the seminormalization, or more precisely we have:

Theorem. *Let S be a Noetherian scheme and M/S be a commutative monoid object in the category of schemes over S where the morphism $M \to S$ is flat, locally of finite type and* AF *(Definition 1.5.26). Then the natural map*

$$\mathrm{Hom}_S((-)^{sn}, M) \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{Hom}_S((-)^{awn}, M) \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{5.0.2}$$

becomes an isomorphism.

The heuristic argument for this theorem to hold is that the extension of scalars makes up for the difference of the degrees of purely inseparable field extensions arising from the maps $T^{awn} \to T$ and $T^{sn} \to T$. Theorem 1.6.7 combined with some elementary number theory and combinatorics makes it possible to transform this somewhat vague idea into a precise proof.

5.1 Representable sheaves and monoid objects

Let S be a Noetherian scheme and suppose that M/S together with $+, 0$ is a commutative monoid object in the category of S-schemes. As discussed in Section E.1 the presheaf $h_{M/S}$ can be considered a commutative monoid object in the category of presheaves on Sch/S and so $h_{M/S}$ can be considered a presheaf of commutative monoids. Furthermore for any Grothendieck topology t we may also consider the t-sheaf $\mathrm{L}_t(M/S)$ as a commutative monoid object in the category of sheaves on the category of Noetherian S-schemes so that $\mathrm{L}_t(M/S)$ may be considered a t-sheaf of commutative monoids.

Letting $\tilde{+} : h_{M/S} \times h_{M/S} \to h_{M/S}$ denote the addition map induced from the commutative monoid object structure on M/S we recall that the addition map $\hat{+} : \mathrm{L}_t(M/S) \times \mathrm{L}_t(M/S) \to \mathrm{L}_t(M/S)$ is the map making the following diagram commutative

$$\begin{tikzcd}
\mathrm{L}_t(M/S) \times \mathrm{L}_t(M/S) \arrow[r, "\hat{+}"] & \mathrm{L}_t(M/S) \\
h_{M/S} \times h_{M/S} \arrow[u] \arrow[r, "\tilde{+}"] & h_{M/S} \arrow[u]
\end{tikzcd} \tag{5.1.1}$$

commutative. Suppose now that M is locally of finite type over S. If the topology t is finer than the uh and coarser than the h topology then we can describe the map $\hat{+}$ in terms of Theorem 4.3.9 as follows: For a given Noetherian S-scheme T and S-morphisms $f_1, f_2 : T^{awn} \to M$ then if $(f_1, f_2) : T^{awn} \to M \times_S M$ denotes the induced map such that $f_i = pr_i \circ (f_1, f_2)$ for $i = 1, 2$ then we have

$$(f_1 + f_2) = + \circ (f_1, f_2) : T^{awn} \to M \underset{S}{\times} M \xrightarrow{+} M. \tag{5.1.2}$$

Similarly if t is finer than the $cd - uh$ topology and coarser than the sd-h topology we can describe the addition using the seminormalization in stead of the absolute weak normalization.

5.2 Auxiliary lemmas

Lemma 5.2.1. *Let $p : T' \to T$ be a universal homeomorphism of S-schemes with T reduced. Suppose that we have an S-morphism $f' : T' \to Y$ and an open affine cover U_i of T such that for every i the morphism $f'|_{p^{-1}(U_i)}$ factors through $p|_{p^{-1}(U_i)}$, then the morphism f' factors through p.*

Proof. By assumption we have S-morphisms $f_i : U_i \to Y$ such that

$$f'|_{p^{-1}(U_i)} = f_i \circ p|_{p^{-1}(U_i)} \tag{5.2.1}$$

for every i. Since p is an epimorphism in the category of S-schemes being a surjective morphism to a reduced scheme, it follows easily that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Thus the f_i glue to give an S-morphism $f : T \to Y$ and it is clear that we have $f' = f \circ p$. □

Lemma 5.2.2. *Let $p : T' \to T$ be a finite universal homeomorphism of Noetherian S-schemes. Then for any $t' \in T'$ with $t = p(t') \in T$ the degree $[k(t') : k(t)]$ of the induced field extension of residue fields $k(t')/k(t)$ is a power of the exponential characteristic of $k(t)$. Furthermore there exists a strictly positive number $d \in \mathbb{N}$ such that for any $t' \in T'$ lying over $t \in T$ the number $[k(t') : k(t)]$ divides d.*

Proof. The first assertion follows from Section 1.4 and Proposition 1.4.17. For the second consider for each number $i \in \mathbb{N}$ the set $U_i := \{t \in T |\, \dim_{k(t)}(p_*\mathcal{O}_{T'})|_t = \dim_{k(t)} \Gamma(T'_t, \mathcal{O}_{T'_t}) < i\}$. It is well known and follows almost immediately from Nakayama's Lemma that these sets are open for every $i \in \mathbb{N}$. Furthermore by quasi-compactness of T there is some $N \in \mathbb{N}$ such that $T = U_N$. For a given prime number p let e_p denote the largest natural number such that $p^{e_p} = [k(t') : k(t)]$ for some point $t' \in T'$ lying over $t \in T$. Note that $p^{e_p} \leq N$ hence the number e_p is well defined and there are only finitely many primes p such that $e_p > 0$, denote these by $p_1, \ldots, p_m$. Now set $d := \prod_{i=1}^m p_i^{e_{p_i}}$. □

Our proof will also involve some elementary number theory. For a prime number p let ν_p denote the p-adic valuation on $\mathbb{Q}$, i.e. if n is a natural number then $\nu_p(n)$ is the exponent of the largest power of p that divides n.

Lemma 5.2.3. *Let p be a prime number and n a natural number. Suppose that $a_1, \ldots, a_k$ are positive natural numbers satisfying $n = \sum_{i=1}^k a_i$. Then we have the inequality*

$$\nu_p(n!) \geq \sum_{i=1}^k \nu_p(a_i!), \tag{5.2.2}$$

and if there exists an $i \in \{1, \ldots, k\}$ such that $p^{\nu_p(n)}$ does not divide a_i then the inequality is strict.

Proof. Recall Legendre's formula:

$$\nu_p(n!) = \sum_{u=1}^{\infty} \left\lfloor \frac{n}{p^u} \right\rfloor, \tag{5.2.3}$$

where there are of course only finitely many non-zero terms in this sum. Whence we have

$$\sum_{i=1}^{k} \nu_p(a_i!) = \sum_{i=1}^{k}\sum_{u=1}^{\infty} \left\lfloor \frac{a_i}{p^u} \right\rfloor = \sum_{u=1}^{\infty}\sum_{i=1}^{k} \left\lfloor \frac{a_i}{p^u} \right\rfloor. \tag{5.2.4}$$

Since for any $u \geq 1$ we have

$$p^u \left(\sum_{i=1}^{k} \left\lfloor \frac{a_i}{p^u} \right\rfloor \right) \leq \sum_{i=1}^{k} a_i = n, \tag{5.2.5}$$

we conclude that

$$\left\lfloor \frac{n}{p^u} \right\rfloor \geq \sum_{i=1}^{k} \left\lfloor \frac{a_i}{p^u} \right\rfloor \tag{5.2.6}$$

for every u thus proving the first claim. Now suppose that there is some $i \in \{1, \dots, k\}$ such that $p^{\nu_p(n)}$ does not divide a_i then $\left\lfloor \frac{a_i}{p^{\nu_p(n)}} \right\rfloor < \frac{a_i}{p^{\nu_p(n)}}$ thus we have

$$\sum_{i=1}^{k} \left\lfloor \frac{a_i}{p^{\nu_p(n)}} \right\rfloor < \sum_{i=1}^{k} \frac{a_i}{p^{\nu_p(n)}} = \frac{n}{p^{\nu_p(n)}} = \left\lfloor \frac{n}{p^{\nu_p(n)}} \right\rfloor \tag{5.2.7}$$

and from (5.2.6) we conclude the claimed strict inequaility. $\square$

5.3 Proof of the theorem

Proof of Theorem 5.0.1. Let T be a seminormal Noetherian scheme over S and consider the natural map

$$h_{M/S}(T) \to \mathrm{L}_t(M/S)(T). \tag{5.3.1}$$

Using the description of the t-sheafification given in Theorem 4.3.9 we recall that this map is given by

$$f \mapsto (f \circ \mu^{(-)^{Perf}}(T)).$$

Since $\mu^{(-)^{Perf}}(T)$ is a surjective morphism to a reduced scheme it is an epimorphism hence the map (5.3.1) is a monomorphism thus by Lemma B.4.1 and Lemma B.2.6 it will remain a monomorphism after we tensorize with $\mathbb{Q}_+$. Hence to complete the proof it is enough to show that if $p : T' \to T$ is a finite universal homeomorphism and we have an S-morphism $f : T' \to (M/S)$ then there exists a natural number $n \in \mathbb{N} \setminus \{0\}$ such that the map $n \cdot f : T' \to (M/S)$ factors through p. To this extent we apply Lemma 5.2.2 to obtain a natural number $d \geq 1$ such that for any $t' \in T'$ lying over $t \in T$ the degree $[k(t') : k(t)]$ of the field extension of residue fields divides d. We now claim that the map $d \cdot f$ factors thorugh p. Note that the map $d \cdot f$ factors as

$$T' \xrightarrow{f} M \xhookrightarrow{\Delta} (M/S)^d \longrightarrow M$$

where the map Δ is the diagonal map and $(M/S)^d \to M$ is given by the monoid structure on M/S. Note that the morphism $(M/S)^d \to M$ is invariant under the action of Σ_d on $(M/S)^d$ hence it factors through $\mathrm{Sym}^d(M/S)$. By Lemma 1.6.13 we can construct an affine cover $\{U_{\alpha,\beta}\}_{\alpha,\beta}$ of M such that $\{(U_{\alpha,\beta}/S_\alpha)^d\}_{\alpha,\beta}$ is an open cover of $(M/S)^d$ and

$$\{\mathrm{Sym}^d(U_{\alpha,\beta}/S_\alpha)\}_{\alpha,\beta}$$

is an open affine cover of $\mathrm{Sym}^d(M/S)$. Note now that $(d\cdot f)|_{f^{-1}(U_{\alpha,\beta})}$ coincides with the following composition

$$f^{-1}(U_{\alpha,\beta}) \xrightarrow{f|_{f^{-1}(U_{\alpha,\beta})}} U_{\alpha,\beta} \xhookrightarrow{\Delta} (U_{\alpha,\beta}/S_\alpha)^d \longrightarrow \mathrm{Sym}^d(U_{\alpha,\beta}/S_\alpha) \longrightarrow M.$$

Thus by Lemma 5.2.1 we can now reduce the proof to proving the following statement:

Let C be a Noetherian seminormal A-algebra and $C \to C'$ an A-algebra homomorphism such that the induced morphism of affine schemes $\mathrm{Spec}(C') \to \mathrm{Spec}(C)$ is a finite universal homeomorphism. Suppose that there exists a strictly positive number $d \in \mathbb{N}$ such that for any $p' \in \mathrm{Spec}(C')$ lying over $p \in \mathrm{Spec}(C)$ the number $[k(p') : k(p)]$ divides d. Then for any flat A-algebra B where we let $m_d : (B/A)^{\otimes d} \to B$ denote the multiplication map induced by $b_1 \otimes \ldots \otimes b_d \mapsto \prod_{i=1}^d b_i$, we have that if $\varphi : B \to C'$ is any A-algebra homomorphism then the composition

$$\mathrm{S}_d(B/A) \longhookrightarrow (B/A)^{\otimes d} \xrightarrow{m_d} B \xrightarrow{\varphi} C',$$

factors through C.

To prove this statement it is enough to show that the image of any generator of the A-algebra $\mathrm{S}_d(B/A)$ in C' is contained in the image of the ring extension $C \to C'$. By Theorem 1.6.7 this A-algebra is generated by the elementary symmetric n-tensors of elements $b \in B$ denoted $\rho_k^d(b)$ and by Remark 1.6.6 the image of $\rho_k^d(b)$ in C' is of the form $\binom{d}{k}\varphi(b)^k = \frac{d!}{k!(d-k)!}\varphi(b)^k \in C'$. Let now $x' \in \mathrm{Spec}(C')$ lie over x in $\mathrm{Spec}(C)$ and suppose that the residue field $k(x)$ has exponential characteristic p. Then note that if $\nu_p(k!(d-k)!) < \nu_p(d!)$ then the image of $\frac{d!}{k!(d-k)!}\varphi(b)^k$ in $k(x')$ will necessarily be 0, and if the image of this element in $k(x')$ is not to be zero then by Lemma 5.2.3 it follows that we must necessarily have some natural number l such that $k = \nu_p(d)l$. But then the image of $\varphi(b)^k$ in $k(x')$ must by the assumption on d and Proposition 1.4.17 be contained in $k(x)$ thus by Proposition 4.1.23 and Theorem 4.1.4 we conclude that the image of $\rho_k^d(b)$ in C' must in fact be contained in the subring C which was what we needed to show. □

Remark 5.3.1. The proof of Theorem 5.0.1 shows that it is enough to tensor with any sub semiring Λ of $\mathbb{Q}_+$ such that every element of $\mathrm{Exp.Char}(S)$ (Definition 3.2.7) is invertible in Λ.

Chapter 6

Chow schemes and Chow monoids

Recall from classical algebraic geometry that there is a variety due to Chow which parametrizes effective cycles of a given dimension and degree on a projective variety. Using a similar construction and a lot of modern theory Suslin-Voevodsky prove in Section 4.4. of [SV00] that if $i : X \to \mathbb{P}^n_S$ is a closed embedding then the presheaf $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ of relative effective cycles of dimension r and degree d with respect to i is representable in the h-topology. This means that there exists some scheme $C_{r,d}$ such that the h-sheafification of the corresponding representable presheaf is isomorphic to the h-sheafification of $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$. On the other hand in [Kol96, Ch. I, Sec. 3,4] one can also find a theory of Chow schemes in mixed characteristic, which in many ways is the same as Suslin-Voevodsky's, yet one could argue that the construction is a little more explicit.

In this chapter we will extract ideas from both [SV00] and [Kol96] and provide a few of our own in order to give a transparent proof of the h-representability of effective relative cycles of fixed dimension and degree. From this we will see that we can form a monoid object $C_r((X,i)/S)$ which we will call the *Chow monoid of degree r cycles with respect to i*. The existence of such a monoid already appears in the literature when S is a field, indeed see for instance [FV00]. We then prove that the sheafification of the Chow monoid with respect to the h topology is isomorphic to the sheafification of the presheaf of relative cycles on X/S with universally integral coefficients. Which we shortly afterwards apply together with Theorem 5.0.1 to finally prove our first main theorem Theorem 6.5.3.

The precise outline of the chapter is as follows: first we recollect some intersection theory such as intersection numbers and the multi-degree of an algebraic cycle on projective space. In the second section we recall the definition and properties of relative effective Cartier divisors. Furthermore Section 6.3 is devoted to proving that the presheaf of effective relative cycles of fixed dimension and degree is representable in the proper/h-topology, which as we mentioned earlier has been proved by Suslin-Voevodsky and our proof incorporates many of their ideas, but is arguably more explicit and detailed than the original proof. After this is done we then introduce the Chow monoid in Section 6.4 which we then use in Section 6.5 to state and prove the first main theorem of the thesis. The final section gives a rough overview of how the first three sections of this chapter compare to the literature.

6.1 Effective relative cycles on multi-projective space

Intersection numbers

Let us now very briefly recall some intersection theory. Let k be a field. For a k-scheme X we let $\mathrm{K}(X)$ denote the Grothendieck group of X and for each natural number r we denote by $\mathrm{K}_r(X)$ the subgroup generated by coherent sheaves whose support has dimension at most r. For a coherent sheaf F on X whose support has dimension at most r one has ([Kol96, VI.2, Corollary 2.3])

$$F \equiv \sum \mathrm{length}_{y_i} F \cdot \mathcal{O}_{Y_i} \pmod{K_{r-1}}$$

where y_i are the generic points of the irreducible components Y_i of $(\mathrm{Supp}(F))_{red}$.

Definition 6.1.1 ([Kol96, Def.VI.2.4])**.** Let L be an invertible sheaf on X. We define an endomorphism of the abelian group $\mathrm{K}(X)$ by the formula

$$\mathrm{c}_1(L) \cdot F = F - L^{-1} \otimes F.$$

Properties of $\mathrm{c}_1(L)$ can be found in [Kol96, VI.2, Prop. 2.5] the most important one for us is that $\mathrm{c}_1(L) \cdot \mathrm{K}_r(X) \subset \mathrm{K}_{r-1}$.

Definition 6.1.2. [Kol96, VI.2, Definition 2.6] Let X be a proper k-scheme and F a coherent sheaf on X. Assume that $m \geq \dim \mathrm{Supp}(F)$ and let $L_1, \dots, L_m$ be invertible sheaves on X. The *intersection number* of $L_1, \dots, L_m$ on F is defined by

$$(L_1 \cdots L_m \cdot F) := \chi(X, \mathrm{c}_1(L_1) \cdots \mathrm{c}_1(L_m) \cdot F)$$

where χ denotes the Euler-characteristic. If $L = L_1 = \ldots = L_m$, then we write $(L^m \cdot F)$ instead of $(L_1 \cdots L_m \cdot F)$. If $Y \subset X$ is a closed subscheme, then we write $(L^m \cdot Y)$ instead of $(L^m \cdot \mathcal{O}_Y)$. To avoid confusion we let $(L)^m$ denote $\mathrm{c}_1(L) \cdots \mathrm{c}_1(L)$ and use $L^{\otimes m}$ to denote the tensor-power. For an effective cycle $\mathcal{Z} = \sum_{i=1}^n a_i z_i$ of dimension at most m we set

$$(L_1 \cdots L_m \cdot \mathcal{Z}) := (L_1 \cdot \ldots L_m \cdot (\oplus_{i=1}^n (\mathcal{O}_{Z_i}^{\oplus a_i})).$$

Intersection numbers are locally constant in families:

Proposition 6.1.3 ([Kol96, VI.2, Proposition 2.9]). *Let $f : X \to S$ be a morphism of schemes, L_i invertible sheaves on X and F a coherent sheaf on X, flat over S, such that* $\mathrm{Supp}(F)$ *is proper over S. Then the function*

$$s \mapsto (L_1 \cdots L_m \cdot (F \otimes \mathcal{O}_{X_s})$$

is locally constant on S.

We also have a projection formula

Proposition 6.1.4 ([Kol96, VI.2, Prop. 2.11]). *Let $f : Y/S \to X/S$ be an k-morphism of proper k-schemes, L_i line bundles on X and F a coherent sheaf on Y. Let $m \geq \dim \mathrm{Supp}(F)$. Then*

$$f^*(L_1) \cdots f^*(L_m) \cdot F = L_1 \cdots L_m \cdot f_*(\mathcal{F}).$$

There are other definitions of intersection numbers. To see the equivalence of Definition 6.1.2 with the one given in [Stacks, Tag 0BEP] simply apply [Kol96, VI.2, Theorem 2.13]. Further in the case of algebraic cycles one can define the intersection number by using the first Chern class (see [Stacks, Tag 02SO] or [Ful98, Ch. 2, Sections 4 and 5] for the definition) and again this coincides with 6.1.2 in the sense of the following lemma:

Lemma 6.1.5 ([Stacks, Tag 0BFI]). *Let k be a field. Let X be a proper scheme over k. Let $Z \subset X$ be a closed subscheme of dimension d. Let $\mathcal{L}_1, \ldots, \mathcal{L}_d$ be invertible $\mathcal{O}_X$-modules. Then*

$$(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z) = \deg(c_1(\mathcal{L}_1) \cap \ldots \cap c_1(\mathcal{L}_1) \cap [Z]_d)$$

Algebraic cycles on multi-projective space

Definition 6.1.6. A *multi-index* $I = (i_1, \ldots, i_s)$ is a sequence of nonnegative integers. For any scheme S we denote by $\mathbb{P}_S^I$ the fiber product

$$\mathbb{P}_S^I := \mathbb{P}_S^{i_1} \underset{S}{\times} \mathbb{P}_S^{i_2} \underset{S}{\times} \ldots \underset{S}{\times} \mathbb{P}_S^{i_s}.$$

We will define the multi-degree of an algebraic cycle on multi-projective space in terms of the following example. The reader not familiar with the topic of intersection theory may instead take the intersection numbers appearing in the next Lemma 6.1.9 as the definition.

Example 6.1.7 ([Ful98, Example 8.3.7])**.** For the multi-index $I = (i_1, \ldots, i_s)$ the exterior product[1]

$$A^*(\mathbb{P}_k^{i_1}) \otimes \ldots \otimes A^*(\mathbb{P}_k^{i_s}) \xrightarrow{\times} A^*(\mathbb{P}_k^I)$$

is an isomorphism and since the Chow group A_r of $\mathbb{P}_k^{i_j}$ is the free $\mathbb{Z}$-module generated by an r-dimensional linear subspace L_r of $\mathbb{P}_k^{i_j}$ it follows that the class of an r-cycle $\mathcal{Z}$ in $A^*(\mathbb{P}_k^I)$ can be written as

$$\mathcal{Z} = \sum d_J \operatorname{cycl}(L_{j_1} \times \ldots \times L_{j_s}) \text{ in } A_r(\mathbb{P}_k^I) \tag{6.1.1}$$

where the sum is over all tuples $J = (j_1, \ldots, j_s)$ of non-negative integers such that $\sum_{l=1}^s j_l = r$.

Definition 6.1.8. Under the assumptions and notations of Example 6.1.7 we shall call the integers d_J of (6.1.1) the *multi-degrees* of the cycle $\mathcal{Z}$.

Lemma 6.1.9. *Let $I = (i_1, \ldots, i_s)$ be a multi-index and let $\mathcal{Z}$ be an r-dimensional cycle on $\mathbb{P}_k^I$. Set $\mathcal{L}_j := pr_j^* \mathcal{O}_{\mathbb{P}_k^{i_j}}(1)$ for $j = 1, \ldots, s$. The multi-degrees of $\mathcal{Z}$ are exactly the intersection numbers of the form*

$$((\mathcal{L}_1)^{r_1} \cdots (\mathcal{L}_s)^{r_s} \cdot \mathcal{Z})$$

whenever $\sum_{l=1}^s r_l = r$ and $r_l \leq i_l$ for all l.

Proof. Write $\mathcal{Z}$ in $A_r(\mathbb{P}_k^I)$ as

$$\sum d_J \operatorname{cycl}(L_{j_1} \times \ldots \times L_{j_s})$$

where the d_J are the multi-degrees of $\mathcal{Z}$.

By additivity of intersection numbers (see [Kol96, VI.2, Proposition 2.7]) and by Lemma 6.1.5 we have that

$$((\mathcal{L}_1)^{r_1} \cdots (\mathcal{L}_s)^{r_s} \cdot \mathcal{Z}) = \sum d_J ((\mathcal{L}_1)^{r_1} \cdots (\mathcal{L}_s)^{r_s} \cdot \operatorname{cycl}(L_{j_1} \times \ldots \times L_{j_s}))$$

By [Ful98, Example 2.5.3] we have

$$((\mathcal{L}_1)^{r_1} \cdots (\mathcal{L}_s)^{r_s} \cdot \operatorname{cycl}(L_{j_1} \times \ldots \times L_{j_s})) = \deg((c_1(\mathcal{O}_{\mathbb{P}_k^{i_1}}(1))^{r_1} \cap L_{j_1})) \times \ldots \times (c_1(\mathcal{O}_{\mathbb{P}_k^{i_s}}(1))^{r_s} \cap L_{j_s})))$$

and since $(c_1(\mathcal{O}_{\mathbb{P}_k^{i_l}}(1))^{r_l} \cap L_{j_l})) = 0$ whenever $r_l > j_l$ and $(c_1(\mathcal{O}_{\mathbb{P}_k^{i_l}}(1))^{j_l} \cap L_{j_l}))$ is the class of a k-point it follows that for sequences $j_1, \ldots, j_s$ and $r_1, \ldots, r_s$ that both sum to r we have

$$((\mathcal{L}_1)^{r_1} \cdots (\mathcal{L}_s)^{r_s} \cdot \operatorname{cycl}(L_{j_1} \times \ldots \times L_{j_s})) = \begin{cases} 1 & \text{if } j_l = r_l \text{ for all } l \\ 0 & \text{otherwise} \end{cases}$$

which completes the proof. □

[1]In this example we use Fultons notation for the Chow rings and groups.

Example 6.1.10. For a multi-index $I = (i_1, \dots, i_s)$ it is well known that the map
$$\mathrm{Pic}(\mathbb{P}_k^{i_1}) \times \dots \times \mathrm{Pic}(\mathbb{P}_k^{i_s}) \to \mathrm{Pic}(\mathbb{P}_k^I)$$
given by
$$(\mathcal{O}_{\mathbb{P}_k^{i_1}}(d_1), \dots, \mathcal{O}_{\mathbb{P}_k^{i_s}}(d_s)) \mapsto \mathcal{O}(d_1, \dots, d_s) := pr_1^* \mathcal{O}_{\mathbb{P}_k^{i_1}}(d_1) \otimes \dots pr_s^* \mathcal{O}_{\mathbb{P}_k^{i_s}}(d_s)$$
is an isomorphism of abelian groups and can for instance be deduced from the isomorphism given by the exterior product in Example 6.1.7. Thus an effective Cartier divisor $\mathcal{Z}$ on $\mathbb{P}_k^I$ is cut out by a global section of $\mathcal{O}(d_1, \dots, d_s)$ where $d_1, \dots, d_s$ are non-negative integers and one can show that the class of the associated $(\sum_{l=1}^s i_l) - 1$ dimensional cycle in $A_{(\sum_{l=1}^s i_l)-1}(\mathbb{P}_k^I)$ coincides with class of
$$\sum_{l=1}^s d_l \,\mathrm{cycl}(\mathbb{P}_k^{i_1} \times \dots \times \mathbb{P}_k^{i_{l-1}} \times H_l \times \mathbb{P}_k^{i_{l+1}} \times \dots \times \mathbb{P}_k^{i_s}).$$
Hence giving an effective cycle of dimension $(\sum_{l=1}^s i_l) - 1$ and with multi-degrees $d_1, \dots, d_s$ is the same as giving a global section of $\mathcal{O}(d_1, \dots, d_s)$ which again coincides with giving a multi-homogeneous polynomial of multi-degree $d_1, \dots, d_s$ i.e. a sum of the form
$$\sum_{j=1}^r a_j \prod_{l=1}^s P_{j,l}$$
where $a_j \in k$ and $P_{j,l}$ is a homogenous polynomial of degree d_l in the coordinates of $\mathbb{P}_k^{i_l}$. Note that the dimension of the global sections of $\mathcal{O}(d_1, \dots, d_s)$ as a k-vector space is equal to $N' = N'(I, D) := \prod_{j=1}^s \binom{i_j + d_j}{d_j}$.

Effective relative cycles on multi-projective space

Definition 6.1.11 ([SV00, p.77]). Let S be a Noetherian scheme. For any multi-index $I = (i_1, \dots, i_k)$ let $\mathcal{Z}$ be an element of $\mathrm{Cycl}^{eff}(\mathbb{P}_S^I/S, r)$. For any point $s \in S$ we denote by $\deg_s(\mathcal{Z})$ the sequence of multi-degrees (after fixing some ordering on these) of the algebraic cycle $\mathcal{Z}_s$ on $\mathbb{P}_{k_s}^I$.

Proposition 6.1.12 ([SV00, Prop. 4.4.8]). *Let S be a Noetherian scheme and $\mathcal{Z}$ an element of $\mathrm{Cycl}^{eff}(\mathbb{P}_S^I/S, r)$. Then the function $s \mapsto \deg_s(\mathcal{Z})$ is locally constant on S.*

Proof. It is sufficient to show that if η is a generic point of S and s is a point in the closure of η then $\deg_\eta(\mathcal{Z}) = \deg_s(\mathcal{Z})$. Since for any cycle $\mathcal{W}$ on $\mathbb{P}_k^I$ and any field extension k'/k we have that the multi-degrees of $\mathcal{W}$ coincides with those of $\mathcal{W} \otimes_k k'$ it is therefore sufficient to show that for some field extensions L, E of k_η, k_s respectively the cycles $\mathcal{Z}_{\mathrm{Spec}(L)}$ and $\mathcal{Z}_{\mathrm{Spec}(E)}$ have the same multi-degree. Let (x_0, x_1, R) be a fat point on S such that the image of

x_1 is $\{\eta, s\}$. Replacing S with $\mathrm{Spec}(R)$ we may assume that S is the spectrum of a discrete valuation ring. In this case $\mathcal{Z} = \sum n_i \,\mathrm{cycl}(Z_i)$ where Z_i are closed subschemes of $\mathbb{P}^I_S$ which are flat and equidimensional over S. Noting that for $s \in S$ the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^{i_l}_S}(1)$ to $\mathbb{P}^{i_l}_{k_s}$ is the line bundle $\mathcal{O}_{\mathbb{P}^{i_l}_{k_s}}(1)$ it follows from Lemma 6.1.9 and Proposition 6.1.3 that multi-degrees are locally constant in flat families which completes the proof. $\square$

Corollary 6.1.13 ([SV00, Cor.4.4.9]). *Let S be a connected Noetherian scheme. Then for any cycle $\mathcal{Z} \in \mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)$ and any point s of S the multi-degree $\deg_s(\mathcal{Z})$ is a sequence of integers which does not depend on s.*

Definition 6.1.14 ([SV00, p.78]). For a Noetherian scheme S, a multi-index $I = (i_1, \ldots, i_k)$ and a sequence of non-negative integers $D = (d_1, \ldots, d_n)$ denote by $\mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}$ the subset in $\mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)_{UI}$ which consists of cycles $\mathcal{Z}$ such that for any point s of S one has $\deg_s(\mathcal{Z}) = D$.

The following lemma is stated without proof in the context of the cdh-topology on p.78 of [SV00].

Lemma 6.1.15. *Under the notations and assumptions of Definition 6.1.14, the presheaf $\mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}$ is a sub-sd-h sheaf of $\mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)_{UI}$ and if S is connected then we have*

$$\mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)_{UI} = \cup_D \,\mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}.$$

Proof. Since multi-degrees of algebraic cycles are invariant under change of base field it is clear that $\mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}$ is a presheaf. Further since we already know that $\mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)_{UI}$ is a sheaf in the sd-h topology it is enough to show that for any element $\mathcal{Z} \in \mathrm{Cycl}^{eff}(\mathbb{P}^I_S/S, r)_{UI}(S)$ such that there exists a sd-h-covering $p : S' \to S$ such that $\mathrm{cycl}(p)(\mathcal{Z}) \in \mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}(S')$ one has $\mathcal{Z} \in \mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S, r)_{UI}(S)$. By surjectivity of sd-h-coverings and the fact that multi-degrees of algebraic cycles are invariant under change of base field the first part of the lemma follows.

The last statement is a direct consequence of Corollary 6.1.13. $\square$

6.2 Relative effective Cartier divisors

Relative effective Cartier divisors

We recall the definition of a relative effective Cartier divisor. We shall follow the definition/convention given in [Stacks] and [Fan+05] which is slightly more general than the one given in [GD67, Section 21.15] in the sense that we do not require the S- scheme X/S to be flat and locally of finite presentation.

Definition 6.2.1. [Stacks, Tag 062T] Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

The following Lemma gives a nice characterisation of relative effective Cartier divisors.

Lemma 6.2.2 ([Fan+05, Lemma 9.3.4])**.** *Let $D \subset X$ be a closed subscheme, $x \in D$ a point, and $s \in S$ its image. Then the following statements are equivalent:*

(1) *The subscheme D is a relative effective Cartier divisor at x (that is, in a neighborhood of x).*

(2) *The schemes X and D are S-flat at x, and the fiber D_s is an effective Cartier divisor on X_s at x.*

(3) *The scheme X is S-flat at x, and the subscheme $D \subset X$ is cut out at x by one element that is regular (a nonzerodivisor) on the fiber X_s.*

Lemma 6.2.3 ([Stacks, Tag 0B8U])**.** *Let $f : X \to S$ be a morphism of schemes. If $D_1, D_2 \subset X$ are relative effective Cartier divisor on X/S then so is $D_1 + D_2$.*

It will be useful for us to keep the following result in mind before moving on.

Lemma 6.2.4 ([Stacks, Tag 0BCN])**.** *Let X be a Noetherian scheme and $D \subset X$ be an effective Cartier divisor. Let $\eta \in D$ be a generic point of an irreducible component of D. Then* $\dim(\mathcal{O}_{X,\eta}) = 1$.

Corollary 6.2.5. *Suppose that $X \to S$ is an equidimensional morphism of dimension r where S is a reduced Noetherian scheme. If D is a relative effective Cartier divisor then* $\mathrm{cycl}_X(D) \in \mathrm{Cycl}^{\mathit{eff}}(X/S, r-1)_{UI}$. *Furthermore we have*

$$\mathrm{cycl}_X(D_1 + D_2) = \mathrm{cycl}_X(D_1) + \mathrm{cycl}_X(D_2) \tag{6.2.1}$$

Proof. If $D \in \mathrm{Div}_{X/S}(S)$ then using for instance Krull's Hauptidealsatz one checks easily that $D \to S$ is an equidimensional morphism of relative dimension $r-1$. The first claim now follows from Corollary 2.2.4. For the latter just note that the set of generic points of $D_1 + D_2$ is the union of the sets of generic points of D_1 and D_2 and all these points are codimension one points of X. By additivity of length (Corollary A.1.9) we conclude the proof. $\square$

Lemma 6.2.6 ([Stacks, Tag 056Q])**.** *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$ where $g' : X' \to X$ is the projection.*

Proposition 6.2.7. *Let $f : X \to S$ be a morphism of schemes and D_1, D_2 be two relative effective Cartier divisors on X/S. Then for any morphisms $g : S' \to S$ we have the following equality*

$$(g')^{-1}(D_1 + D_2) = (g')^{-1}(D_1) + (g')^{-1}(D_2).$$

In otherwords pullback yields a presheaf of abelian groups

$$\mathrm{Div}_{X/S} : (\mathrm{Sch})^{op} \to \mathrm{Ab}\,.$$

Proof. This result essentially follows from [Vak13, Ex.24.3.H]. □

Lemma 6.2.8. *Let $f : X \to S$ be an equidimensional morphism of dimension r and let $Z = \overline{\{z\}}$ be an integral subscheme of X with z lying over a generic point η of S such that $\dim Z_\eta = r - 1$. Then z is a codimension 1 point of X.*

Proof. From the equality

$$\dim(X_\eta) = \dim(\mathcal{O}_{X_\eta,z}) + \dim(Z_\eta) \tag{6.2.2}$$

we see that $\dim(\mathcal{O}_{X_\eta,z}) = 1$. Moreover since z is over the generic point η we have

$$\mathcal{O}_{X_{\mathrm{red}},z} = \mathcal{O}_{(X_\eta)_{\mathrm{red}},z} \tag{6.2.3}$$

proving that z is indeed a codimension one point of X. □

Comparison with effective relative cycles

Lemma 6.2.9. *Let S be a reduced Noetherian scheme and $X \to S$ a smooth equidimensional morphism of dimension r and D a relative effective Cartier divisor on X/S. Then D does not have any embedded components.*

Proof. Since D is flat over S all associated points of D are mapped to associated points of S which are necessarily generic points since S is reduced. If D has an embedded point mapping to a generic point $\eta \in S$ then there is also a generic point of D lying over η. As association of points (of a ring or module) commutes with localization it is clear that the fiber D_η must have an embedded point. This contradicts Lemma 6.2.2 Item (2) which tells us that that D_η is an effective Cartier divisor on the regular scheme X_S hence D_η must necessarily be Cohen-Macaulay and can thus not have any embedded components. □

Corollary 6.2.10. *Let S be a reduced Noetherian scheme and $X \to S$ a smooth equidimensional morphism of dimension r. If $\mathcal{Z} \in \mathrm{Cycl}^{\mathit{eff}}(X/S, r-1)_{\mathbb{Q}_+}(S)$ then there is at most one relative effective Cartier D on X/S such that $\mathrm{cycl}_X(D) = \mathcal{Z}$.*

Proof. This follows from Lemma 6.2.9, Lemma 6.2.4 and Proposition 1.7.10. □

Proposition 6.2.11 ([SV00, Prop. 3.4.8])**.** *Let S be a normal Noetherian scheme and $X \to S$ a smooth equidimensional morphism of dimension r. Then the* cycl *function induces a well defined isomorphism of monoids*

$$\begin{aligned} &\mathrm{Div}_{X/S}(S) \to \mathrm{Cycl}^{eff}(X/S, r-1)_{UI}(S) \\ &D \mapsto \mathrm{cycl}_X(D) \end{aligned}$$

Moreover by restricting both presheaves $\mathrm{Div}_{X/S}$ and $\mathrm{Cycl}^{eff}(X/S, r-1)_{UI}$ to the category of normal Noetherian schemes the aforementioned map gives a natural transformation.

Proof. The fact that the cycl function induces a homomorphism of monoids

$$\mathrm{Div}_{X/S}(S) \to \mathrm{Cycl}^{eff}(X/S, r-1)_{UI}(S)$$

is the content of Corollary 6.2.5

Let now z be any point of X lying over a generic point of S such that the closure Z is equidimensional of relative dimension $r-1$ over S. By Lemma 6.2.8 it follows that z is a codimension 1-point. Furthermore [GD67, Prop. 21.14.3] says (among other things) that any codimension one cycle Z of X such that $\mathrm{Supp}(Z)$ does not contain any irreducible components of any fiber $f^{-1}(s) = X_s$, is locally principal and since a smooth scheme over a normal scheme is normal ([Stacks, Tag 07TD]) it follows from Corollary 1.7.11 that Z is an effective Cartier divisor. If $x \in Z$ is a point of Z then since $X_{f(x)}$ is smooth it is clear that the element which cuts out Z at the point x is regular (nonzero divisor) on the fiber $X_{f(x)}$ thus by Lemma 6.2.2.(3) it follows that Z is a relative effective cartier divisor. This proves that the map $\mathrm{Div}_{X/S} \to \mathrm{Cycl}^{eff}_{equi}(X/S, r-1)_{UI} = \mathrm{Cycl}^{eff}(X/S, r-1)_{UI}$, the last equality following from 2.1.23, is surjective and it is injective by Corollary 6.2.10. The final statement trivially follows from Lemma 2.3.19. □

Proposition 6.2.12. *Let S be a Noetherian scheme and $X \to S$ be a smooth equidimensional morphism of dimension r. Suppose that $Z \subset X$ is a closed integral subscheme of X flat and equidimensional of dimension $r-1$ over S. Then Z is a relative effective Cartier divisor on X/S.*

Proof. By Lemma 6.2.2 it is enough to prove that Z_s is an effective Cartier divisor on X_s for any point $s \in S$. Let $\eta \in S$ be the image of the generic point of Z. For an arbitrary point s we apply Corollary 2.1.3 to find a discrete valuation ring R with field of fractions $k(\eta)$ and a morphism $\mathrm{Spec}(R) \to S$ taking the closed point to s and the generic point to η. By [Stacks, Tag 036D] (or [Gro71, Expose II, Proposition 3.1]) it follows that $X_R = X \times_S \mathrm{Spec}(R)$ is a regular scheme. Moreover since Z is an integral scheme so is Z_η and as Z is flat over S we see that $Z_R = Z \times_S \mathrm{Spec}(R)$ has no embedded components. Furthermore from Lemma 6.2.8 it follows that Z_R is an integral codimension

1 subscheme of X_R. Thus we can apply Corollary 1.7.11 to see that Z_R is a relative effective Cartier divisor which implies that the special fiber of Z_R is an effective Cartier divisor. By Lemma 1.7.12 we conclude that Z_s is an effective Cartier divisor which is what we needed to show.

□

Remark 6.2.13. If the morphism $X \to S$ in Proposition 6.2.12 is also projective, then the result follows directly from [Kol96, Thm. 1.13]. In fact our proof is very similar to the proof of the second statement of loc.cit.

Corollary 6.2.14. *Let S be a Noetherian scheme and $X \to S$ be a smooth equidimensional morphism of dimension r. For any relative cycle*

$$\mathcal{Z} \in \mathrm{Cycl}^{\mathit{eff}}(X/S, r-1)_{UI}$$

there is a proper surjective morphism $p : S' \to S$ (depending on $\mathcal{Z}$) from a reduced Noetherian scheme S' and a relative effective Cartier divisor D on $S' \times_S X/S'$ such that

$$\mathrm{cycl}_{S' \times_S X}(D) = \mathrm{cycl}(p)(\mathcal{Z}). \tag{6.2.4}$$

Proof. For $\mathcal{Z} = \sum n_i z_i \in \mathrm{Cycl}(X/S, r-1)$ let Z_i denote the closure of z_i in X. By Theorem 1.2.3 we can find a morphism $p : S' \to S$ that is a blow up of S_{red} such that the proper transforms $\tilde{Z}_i$ of Z_i are flat of equidimension $r-1$ over S' and by Lemma 2.3.14 we have

$$\mathrm{cycl}(p)(\mathcal{Z}) = \sum n_i \,\mathrm{cycl}_{X \times_S S'}(\tilde{Z}_i). \tag{6.2.5}$$

By Proposition 6.2.12 we have that the $\tilde{Z}_i$ are relative effective Cartier divisors on $S' \times_S X/S'$ and by Corollary 6.2.5 we have

$$\mathrm{cycl}_{S' \times_S X}(\sum n_i \tilde{Z}_i) = \mathrm{cycl}(p)(\mathcal{Z}). \tag{6.2.6}$$

□

The following Lemma is trivial and well known.

Lemma 6.2.15. *Suppose that* t *is a Grothendieck topology where the reduced-induced subscheme structure $\{X_{red} \to X\}$ is a covering (for any scheme X). Then for any t-sheaf $\mathcal{F}$ the map $\mathcal{F}(X) \to \mathcal{F}(X_{red})$ is an isomorphism.*

Proof. Just note that we have $(X_{red} \times_X X_{red}) = X_{red}$ and the result follows immediately from the sheaf sequence. □

The following result is implicit in [SV00, Lemma 4.4.10].

Theorem 6.2.16. *Let S be a Noetherian scheme and $X \to S$ be a smooth equidimensional morphism of dimension r. Then we have an isomorphism of h-sheaves on the category of Noetherian schemes*

$$(\mathrm{Div}_{X/S})_h \cong (\mathrm{Cycl}^{\mathit{eff}}(X/S, r-1)_{UI})_h.$$

Proof. By Lemma 6.2.15 it is enough to show that the morphism

$$\mathrm{cycl} \circ (-)_{\mathrm{red}} : \mathrm{Div}_{X/S} \circ (-)_{\mathrm{red}} \to \mathrm{Cycl}^{\mathit{eff}}(X/S, r-1)_{UI} \circ (-)_{\mathrm{red}} \tag{6.2.7}$$

is an h-local isomorphism. By Corollary 6.2.10 it is already a monomorphism and by Corollary 6.2.14 it is h-locally an epimorphism. $\square$

Remark 6.2.17. The proof will work for any topology containing the proper topology.

Representability of relative effective Cartier divisors in projective space

The so called Cohomology and base change theorem states that in good circumstances, given a fibered diagram:

$$\begin{array}{ccc} W & \xrightarrow{\psi'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\psi} & Y, \end{array}$$

and a coherent sheaf $\mathcal{F}$ on X, the natural map:

$$\phi_Z^i : \psi^*(\mathrm{R}^i\, \pi_* \mathcal{F}) \to \mathrm{R}^i(\pi')_*(\psi')^*(\mathcal{F})$$

is an isomorphism. We recall the precise statement whose proof can be found in [GD63] or [Vak13].

Theorem 6.2.18 (Grothendieck)**.** *Suppose $\pi : X \to Y$ is proper morphism of finite presentation, $\mathcal{F}$ is coherent and flat over Y, and assume that for a given integer i and point $y \in Y$ the natural base change morphism ϕ_y^i : $(\mathrm{R}^i\, \pi_* \mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to \mathrm{H}^i(X_y, \mathcal{F}_y)$ is surjective. Then the following hold:*

(1) *There is an open neighborhood U of y such that for any $\psi : Z \to U$, the natural base change morphism*

$$\phi_Z^i : \psi^*(\mathrm{R}^i\, \pi_* \mathcal{F}) \to \mathrm{R}^i(\pi')_*(\psi')^*(\mathcal{F})$$

is an isomorphism. In particular, ϕ_y^i is an isomorphism.

(2) *Furthermore,* ϕ_y^{i-1} *is surjective (hence an isomorphism by (1)) if and only if* $\mathrm{R}^i\,\pi_*\mathcal{F}$ *is locally free in some neighborhood of* y *(or equivalently* $(\mathrm{R}^i\,\pi_*\mathcal{F})_y$ *is a locally free* $\mathcal{O}_{Y,y}$*-module). This in turn implies that* h^i *is constant in a neighborhood of* y*.*

This is a powerful theorem used in the construction of many moduli spaces. A first simple consequence is the following corollary:

Corollary 6.2.19. *Under the assumptions of Theorem 6.2.18, if* $\mathrm{H}^i(X_y, \mathcal{F}_y) = 0$ *for some* $y \in Y$ *then there is an open neighborhood* U *of* y *such that*

(1) $(\mathrm{R}^i\,\pi_*\mathcal{F})|_U = 0$ *and*

(2) $\mathrm{H}^i(X_{y'}, \mathcal{F}_{y'}) = 0$ *for all* $y' \in U$*.*

In the case $i = 1$ *there is some open neighborhood* V *of* y *such that* $(\pi_*\mathcal{F})|_V$ *is a locally free sheaf and* $\phi_{y'}^0 : (\pi_*\mathcal{F})_{y'} \otimes_{\mathcal{O}_{Y,y'}} k(y') \to \mathrm{H}^i(X_{y'}, \mathcal{F}_{y'})$ *is an isomorphism with constant* h^i *in a neighborhood of* y*.*

Proof. By (1) of Theorem 6.2.18 it follows that $(\mathrm{R}^i\,\pi_*\mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) = 0$ and thus by Nakayama's Lemma there is some neighborhood U such that $(\mathrm{R}^i\,\pi_*\mathcal{F})|_U = 0$. By taking $\psi : Z \to U$ to be the canonical map $\mathrm{Spec}(k(y')) \to U$ for any point $y' \in U$ it follows from what we have already proved that the source of ϕ_Z^i is zero and hence also the target, but by applying the target sheaf to $\mathrm{Spec}(k(y'))$ one gets the cohomology group $\mathrm{H}^i(X_{y'}, \mathcal{F}_{y'})$ thus proving part (2) as well.

For the last statements note that by what we have proved in part (1) we have that $(\mathrm{R}^1\,\pi_*\mathcal{F})|_U$ is locally free (of rank 0) which by part (2) of Theorem 6.2.18 implies that ϕ_y^0 is an isomorphism and we can now apply Theorem 6.2.18 in the case of $i = 0$. Since ϕ_y^{-1} is obviously surjective as it is a map of trivial modules it follows from part (2) of Theorem 6.2.18 that $\mathrm{R}^0\,\pi_*\mathcal{F} \cong \pi_*\mathcal{F}$ is locally free in some neighborhood V of y and this in turn implies that h^0 is constant in a neighborhood of y. Hence the final statement also follows. □

Recall from Example 6.1.10 that given a multi-index $I = (i_1, \ldots, i_k)$ and K a field then the effective Cartier divisors on $\mathbb{P}_K^I$ are cut out by a multi-homogeneous polynomial of multi-degree $d_1, \ldots, d_k$ for non-negative integers $d_1, \ldots, d_k$.

Definition 6.2.20. For fixed sequences of positive integers $I = (i_1, \ldots, i_k)$ and $D = (d_1, \ldots, d_k)$ let $\mathrm{H}_{D,I}$ be the subpresheaf of $\mathrm{Div}_{\mathbb{P}_S^I/S}$ consisting of those relative effective cartier divisors which are of finite presentation over S and whose fibers have multi-degree D.

The following proposition is the multi-projective version of [Vak13, Proposition 28.3.6].

Proposition 6.2.21. *For a multi-index $I = (i_1, \dots, i_s)$ and sequence of non-negative integers $D = (d_1, \dots, d_s)$ set $N = N(I, D) := N' - 1 = N'(I, D) - 1 = \prod_{j=1}^{s} \binom{i_j + d_j}{d_j} - 1$. Then the presheaf $\mathrm{H}_{D,I}$ is represented by the scheme $\mathbb{P}^N_S$.*

Proof. It is enough to prove it in the case $S = \mathrm{Spec}(\mathbb{Z})$. Let $\mathcal{P}$ be the subset of $M = \Gamma(\mathbb{P}^I_{\mathbb{Z}}, \mathcal{O}(d_1, \dots, d_s))$ consisting of products of elements of the form $p = \prod_{j=1}^{s} P_j$ where P_j is a monomial of degree d_j in the variables on $\mathbb{P}^{i_j}_{\mathbb{Z}}$ say $(x_{i_j})_0, \dots, (x_{i_j})_{i_j}$. Note that $\mathcal{P}$ forms a basis for M as a $\mathbb{Z}$-module and that $|\mathcal{P}| = N'$. Hence we may use the $p \in \mathcal{P}$ to index the coordinates on $\mathbb{P}^N_{\mathbb{Z}}$ and we denote them by y_p. Consider now the closed subscheme

$$\mathscr{X} = V(\sum_{p \in \mathcal{P}} y_p \cdot p) \subset \mathbb{P}^N_{\mathbb{Z}} \times \mathbb{P}^I_{\mathbb{Z}}.$$

Note that $\mathscr{X}$ is clearly finitely presented and the fibers are hypersurfaces in $\mathbb{P}^I$ cut out by multi-homogeneous polynomials of multi-degree D. One can check flatness in several ways applying part (3) of Lemma 6.2.2 being one of them. Thus $\mathscr{X} \in \mathrm{H}_{D,I}(\mathbb{P}^N_{\mathbb{Z}})$ and by Yoneda lemma we have a map $h_{\mathbb{P}^N_{\mathbb{Z}}} \to \mathrm{H}_{D,I}$. We will show that this is an isomorphism.

Let S be any scheme and let $X \in \mathrm{H}_{D,I}(S)$. Consider the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^I_{\mathbb{Z}}} \to \mathcal{O}_X \to 0$$

and tensorise this with the invertible sheaf $\mathcal{O}(d_1, \dots, d_s)$ which we denote by

$$0 \to \mathcal{I}_X(d_1, \dots, d_s) \to \mathcal{O}(d_1, \dots, d_s) \to \mathcal{O}_X(d_1, \dots, d_s) \to 0.$$

For any point $t \in S$ note that we have $\mathcal{I}(d_1, \dots, d_s)_t = \mathcal{O}_{\mathbb{P}^I_{k_t}}$ and by for instance applying the Künneth formula we see that $\mathrm{H}^1(\mathbb{P}^I_{k_t}, \mathcal{O}_{\mathbb{P}^I_{k_t}}) = 0$. Thus we can apply Corollary 6.2.19 and we get that $(pr_S)_*(\mathcal{I}_X(d_1, \dots, d_s))$ is a locally free sheaf and we have an exact sequence

$$0 \to (pr_S)_*\mathcal{I}_X(d_1, \dots, d_s) \to (pr_S)_*\mathcal{O}(d_1, \dots, d_s) \to (pr_S)_*\mathcal{O}_X(d_1, \dots, d_s) \to 0.$$

Further $\mathrm{h}^0(\mathbb{P}^I_{k_t}, \mathcal{I}(d_1, \dots, d_s)_t) = \mathrm{h}^0(\mathbb{P}^I_{k_t}, \mathcal{O}_{\mathbb{P}^I_{k_t}}) = 1$ where we have used the Künneth formula to compute the last cohomology group. Hence from Corollary 6.2.19 it follows that the rank of the locally free sheaf $(pr_S)_*(\mathcal{I}_X(d_1, \dots, d_s))$ is equal to 1 thus $(pr_S)_*(\mathcal{I}_X(d_1, \dots, d_s))$ is a line bundle on S. Note also that $(pr_S)_*\mathcal{O}(d_1, \dots, d_s)$ is a free $\mathcal{O}_S$-module of rank N' and that on any affine open $\mathrm{Spec}(B)$ of S we have that $(pr_S)_*\mathcal{O}(d_1, \dots, d_s)(\mathrm{Spec}(B))$ is isomorphic to the B-module of multi-homogeneous polynomials of multi-degree $D = (d_1, \dots, d_s)$ with coefficients in B. Further since $(pr_S)_*(\mathcal{I}_X(d_1, \dots, d_s))$ is a line bundle it follows that we can cover S by affine opens $\mathrm{Spec}(B_i)$ such that the image of the map

$$(pr_S)_*(\mathcal{I}_X(d_1, \dots, d_s)(\mathrm{Spec}(B_i)) \to (pr_S)_*(\mathcal{O}(d_1, \dots, d_s))(\mathrm{Spec}(B_i))$$

is generated by a multi-homogeneous polynomial F_i which cuts X out over $\mathrm{Spec}(B_i)$. Furthermore since all fibers of X are hypersurfaces of multi-degee D it follows that F_i cannot vanish at any point of $\mathrm{Spec}(B_i)$ and thus the coefficients of F_i generate the unit ideal of B_i. From this we easily deduce that we have a surjection

$$((pr_S)_*\mathcal{O}(d_1,\dots,d_s))^\vee \cong \mathcal{O}_S^{N'} \to (pr_S)_*(\mathcal{I}_X(d_1,\dots,d_s)^\vee := \mathcal{L}$$

which means that there are N' global sections which we denote by f_p for each $p \in \mathcal{P}$ on the line bundle $\mathcal{L}$ with no common zeros. Hence by the universal property of $\mathbb{P}^N_{\mathbb{Z}}$ we get a morphism of schemes

$$f_X : S \to \mathbb{P}^N_{\mathbb{Z}}.$$

Note that on an affine open $\mathrm{Spec}(B)$ of S where $(pr_S)_*\mathcal{O}(d_1,\dots,d_s)$ trivialises and the image of $(pr_S)_*\mathcal{I}_X(d_1,\dots,d_s) \to (pr_S)_*\mathcal{O}(d_1,\dots,d_s)$ is generated by a multi-homogeneous polynomial of multi-degree D of the form $F = \sum_{p\in\mathcal{P}} a_p \cdot p$, the element $(f_p)|_{\mathrm{Spec}\,B} \in \mathcal{L}(\mathrm{Spec}(B))$ corresponds to the $(\mathcal{O}_S)|_{\mathrm{Spec}(B)}$ -module map $(\mathcal{O}_S)|_{\mathrm{Spec}\,B} \to (\mathcal{O}_S)|_{\mathrm{Spec}\,B}$ given by multiplication with a_p. From this description it is clear that we have

$$f_X^*\mathscr{X} = X$$

proving surjectivity of the map $h_{\mathbb{P}^N_{\mathbb{Z}}} \to \mathrm{H}_{D,I}$ and injectivity is rather clear. □

Remark 6.2.22. Proposition 6.2.21 is a special case of [Kol96, Ex. I.1.14.2].

Corollary 6.2.23. *For a multi-index $I = (i_1,\dots,i_k)$ and a sequence of non-negative integers $D = (d_1,\dots,d_k)$ let $\mathbb{P}_S^{N(I,D)}$ be the projective space of Proposition 6.2.21 representing the presheaf $\mathrm{H}_{D,I}$ and let $\mathrm{H}^{irr}_{D,I}$ denote the sub-presheaf of $\mathrm{H}_{D,I}$ consisting of those $X \to S \in \mathrm{H}_{D,I}(S)$ such that for any geometric point $x : \mathrm{Spec}(k) \to S$ the fiber x^*X is an integral hypersurface in $\mathbb{P}^I_k$, (i.e. the polynomial cutting out x^*X is irreducible). There exists an open subset $U_{D,I} \subset \mathbb{P}_S^{N(I,D)}$ representing $\mathrm{H}^{irr}_{D,I}$.*

Proof. Note that the universal hypersurface

$$\mathscr{X} \subset \mathbb{P}_S^{N(I,D)} \times_S \mathbb{P}^I_S \to \mathbb{P}_S^{N(I,D)}$$

is both proper and flat over $\mathbb{P}_S^{N(I,D)}$, hence by [GD67, (12.2.1)] the set

$$U_{D,I} := \{t \in \mathbb{P}_S^{N(I,D)} \mid \mathscr{X}_t \text{ is geometrically integral }\}$$

is an open subset of $\mathbb{P}_S^{N(I,D)}$ and it is clear that $U_{D,I}$ represents $\mathrm{H}^{irr}_{D,I}$.

□

The monoid of equi-degree hypersurfaces

Let $n, r \in \mathbb{N}$ be given. For $d \geq 0$ we let $\mathrm{H}_{\underline{d},\underline{n}}$ denote the presheaf $\mathrm{H}_{D,I}$ (Definition 6.2.20) where D (resp. I) is the $(r+1)$-tuple where every entry is equal to the integer d (resp. n). We will define $\mathrm{H}_{\underline{0},\underline{n}}$ to be the terminal object in the category of presheaves. Similarly we shall also denote by ${\mathrm{H}^{irr}}_{\underline{d},\underline{n}}$ the presheaf $\mathrm{H}^{irr}_{D,I}$. By Proposition 6.2.21 we know that the presheaf $\mathrm{H}_{\underline{d},\underline{n}}$ is representable by a scheme which we shall from now on denote by $\mathbb{H}_{\underline{d},\underline{n}}$ and the presheaf ${\mathrm{H}^{irr}}_{\underline{d},\underline{n}}$ is represented by an open subscheme of $\mathbb{H}_{\underline{d},\underline{n}}$ which we shall denote by ${\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$.

For any pair $d_1, d_2 \in \mathbb{N}$ we have a natural transformation

$$\alpha^{d_1,d_2} : \mathrm{H}_{\underline{d_1},\underline{n}} \times \mathrm{H}_{\underline{d_2},\underline{n}} \to \mathrm{H}_{\underline{d_1+d_2},\underline{n}} \tag{6.2.8}$$

given by sending the pair of relative effective Cartier divisors (D_1, D_2) to their sum $D_1 + D_2$. Commutativity and associativity of addition ensures that for $d_1, d_2, d_3 \in \mathbb{N}$ the following diagrams commute:

1.

$$\begin{tikzcd} \mathrm{H}_{\underline{d_1},\underline{n}} \times \mathrm{H}_{\underline{d_2},\underline{n}} \arrow[r, "\alpha^{d_1,d_2}"] \arrow[d, "swap"] & \mathrm{H}_{\underline{d_1+d_2},\underline{n}} \arrow[d, equal] \\ \mathrm{H}_{\underline{d_2},\underline{n}} \times \mathrm{H}_{\underline{d_1},\underline{n}} \arrow[r, "\alpha^{d_2,d_1}"] & \mathrm{H}_{\underline{d_1+d_2},\underline{n}}. \end{tikzcd} \tag{6.2.9}$$

2.

$$\begin{tikzcd} \mathrm{H}_{\underline{d_3},\underline{n}} \times \mathrm{H}_{\underline{d_1},\underline{n}} \times \mathrm{H}_{\underline{d_2},\underline{n}} \arrow[r, "\mathrm{H}_{\underline{d_3},\underline{n}} \times \alpha^{d_1,d_2}"] \arrow[d, "\alpha^{d_3,d_1} \times \mathrm{H}_{\underline{d_2},\underline{n}}"] & \mathrm{H}_{\underline{d_3},\underline{n}} \times \mathrm{H}_{\underline{d_1+d_2},\underline{n}} \arrow[d, "\alpha^{d_3,d_1+d_2}"] \\ \mathrm{H}_{\underline{d_3+d_1},\underline{n}} \times \mathrm{H}_{\underline{d_2},\underline{n}} \arrow[r, "\alpha^{d_3+d_1,d_2}"] & \mathrm{H}_{\underline{d_1+d_2+d_3},\underline{n}}. \end{tikzcd} \tag{6.2.10}$$

3.

$$\begin{tikzcd} \mathrm{H}_{\underline{d_1},\underline{n}} \arrow[r, "\cong"'] \arrow[rr, bend left, equal] & \mathrm{H}_{\underline{0},\underline{n}} \times \mathrm{H}_{\underline{d_1},\underline{n}} \arrow[r, "\alpha^{0,d_1}"'] & \mathrm{H}_{\underline{d_1},\underline{n}}. \end{tikzcd} \tag{6.2.11}$$

From Yoneda Lemma we get induced morphisms of schemes

$$\beta^{d_1,d_2} : \mathbb{H}_{\underline{d_1},\underline{n}} \underset{S}{\times} \mathbb{H}_{\underline{d_2},\underline{n}} \to \mathbb{H}_{\underline{d_1+d_2},\underline{n}}. \tag{6.2.12}$$

which will play a role in proving h-representability of relative cycles.

For later use note that we can apply Construction E.2.3 to get a graded commutative monoid object in the category of schemes

$$\mathbb{H}_{r,n} := \coprod_{d \geq 0} \mathbb{H}_{\underline{d},\underline{n}}, \ \beta : \mathbb{H}_{r,n} \underset{S}{\times} \mathbb{H}_{r,n} \to \mathbb{H}_{r,n}. \tag{6.2.13}$$

Observation 6.2.24. For a positive integer m we can add a relative effective Cartier divisor of multi-degree $(d, d, \dots, d)$ with itself m times to get a hypersurface of multi-degree $(md, md, \dots, md)$. This corresponds to a map of schemes

$$m \cdot \beta : (\mathbb{H}_{\underline{d},\underline{n}}/S)^m \to \mathbb{H}_{\underline{m \cdot d},\underline{n}} \tag{6.2.14}$$

which we can explicitly describe as follows: We have a commutative diagram

$$\begin{tikzcd}
(\mathrm{H}_{\underline{d},\underline{n}})^m \arrow[r] & \mathrm{H}_{\underline{m\cdot d},\underline{n}} \\
(h_{\mathrm{H}_{\underline{d},\underline{n}}})^m \arrow[u, "\cong"] & \\
h_{(\mathbb{H}_{\underline{d},\underline{n}}/S)^m} \arrow[u, "\cong"] \arrow[r, "h(m\cdot\beta)"] & h_{\mathbb{H}_{\underline{m\cdot d},\underline{n}}} \arrow[uu, "\cong"']
\end{tikzcd}$$

from which we see that if we let $G := ((\mathbb{P}^n_S))^{r+1}$ and let $\mathscr{X}_d$ denote the universal hypersurface on $\mathbb{H}_{\underline{d},\underline{n}} \times_S G$, then the morphism $m \cdot \beta : (\mathbb{H}_{\underline{d},\underline{n}}/S)^m \to \mathbb{H}_{\underline{m\cdot d},\underline{n}}$ corresponds to the relative effective Cartier divisor $\sum_{i=1}^m pr_i^* \mathscr{X}_d$ on $(\mathbb{H}_{\underline{d},\underline{n}}/S)^m \times_S G$. Suppose now first that $S = \mathrm{Spec}(\mathbb{Z})$.

If for each $j = 1, \dots, r+1$ we give $\mathbb{P}^n_{\mathbb{Z}}$ coordinates $((x_j)_0, \dots, (x_j)_n$ we let $\mathcal{P}(d)$ be the subset of $M = \Gamma(G, \mathcal{O}(d, \dots, d))$ consisting of products of elements of the form $p = \prod_{j=1}^{r+1} P_j$ where P_j is a monomial of degree d in the coordinates of the j'th copy of $\mathbb{P}^n_{\mathbb{Z}}$. Note that $\mathcal{P}(d)$ forms a basis for M as a $\mathbb{Z}$-module and from the proof of Proposition 6.2.21 we know that the scheme $\mathbb{H}_{\underline{d},\underline{n}}$ is a projective space over $\mathrm{Spec}(\mathbb{Z})$ whose coordinates are in bijection with $\mathcal{P}(d)$, hence we can denote them as $\{y_p\}_{p\in\mathcal{P}(d)}$. We then compute that the relative effective Cartier divisor $\sum_{i=1}^m pr_i^* \mathscr{X}_d$ on $(\mathbb{H}_{\underline{d},\underline{n}}/S)^m \times_S G$ is cut out by the following multinomial

$$\sum_{p\in\mathcal{P}(m\cdot d)} \left(\sum_{\substack{(p_1,\dots,p_m)\in\mathcal{P}(d)^m \\ \prod_{i=1}^m p_i = p}} \otimes_{i=1}^m pr_i^*(y_{p_i}) \right) \cdot p \in$$

$$\Gamma\left((\mathbb{H}_{\underline{d},\underline{n}}/\mathrm{Spec}(\mathbb{Z}))^m \times G, \left(\bigotimes_{i=1}^m pr_i^*(\mathcal{O}_{\mathbb{H}_{\underline{d},\underline{n}}}(1)) \right) \boxtimes \mathcal{O}_G(m\cdot d, \dots, m\cdot d) \right).$$

Thus the global sections of the line bundle $\left(\bigotimes_{i=1}^m pr_i^*(\mathcal{O}_{\mathbb{H}_{\underline{d},\underline{n}}}(1))\right)$ corresponding to the morphism $m \cdot \beta : (\mathbb{H}_{\underline{d},\underline{n}}/S)^m \to \mathbb{H}_{\underline{m\cdot d},\underline{n}}$ are the elements of the following set

$$\left\{ \left(\sum_{\substack{(p_1,\dots,p_m)\in\mathcal{P}(d)^m \\ \prod_{i=1}^m p_i = p}} \otimes_{i=1}^m pr_i^*(y_{p_i}) \right) \right\}_{p\in\mathcal{P}(m\cdot d)}. \tag{6.2.15}$$

This solves the problem when working over $\mathrm{Spec}(\mathbb{Z})$. The general case is derived from this using base change.

6.3 Proper representability of relative cycles on projective space

The Chow homomorphism

For a given $r \in \mathbb{N}$ consider for each $j = 1, \dots, r+1$ a copy of $(\mathbb{P}^n_{\mathbb{Z}})^\vee \cong \mathbb{P}^n_{\mathbb{Z}}$ with coordinates $y_{0,j}, y_{1,j}, \dots, y_{n,j}$ and also consider an additional $\mathbb{P}^n_{\mathbb{Z}}$ with coordinates $x_0, \dots, x_n$. Denote by G the product

$$G := (\mathbb{P}^n_{\mathbb{Z}})^\vee \underset{\mathbb{Z}}{\times} \times \dots \underset{\mathbb{Z}}{\times} (\mathbb{P}^n_{\mathbb{Z}})^\vee$$

of the $r+1$-copies of (dual) projective spaces. Furthermore for each $j = 1, \dots, r+1$ let

$$f_j := \sum_{i=0}^{n} x_i y_{i,j} \in \Gamma(\mathbb{P}^n_{\mathbb{Z}} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} G, pr_1^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) \otimes pr_j^* \mathcal{O}_{(\mathbb{P}^n_{\mathbb{Z}})^\vee}(1))$$

and set

$$L := \cap (V_{f_j}) \lhook\joinrel\longrightarrow \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} G.$$

Note that the fibers of the projection $f : L \to \mathbb{P}^n_{\mathbb{Z}}$ onto the first factor are all isomorphic to $(\mathbb{P}^{n-1})^{r+1}$ over the appropriate field, and thus for a k-point $x : \mathrm{Spec}(k) \to \mathbb{P}^n_{\mathbb{Z}}$ the k-points of the fiber over x may be considered as tuples $(H_1, \dots, H_{r+1})$ where H_j are hyperplanes in $\mathbb{P}^n_k$ that all contain the point x, hence the notation $(\mathbb{P}^n_{\mathbb{Z}})^\vee$ used in the construction of G. The following statement can be found without proof on p.78 of [SV00].

Lemma 6.3.1. *The projection $f : L \to \mathbb{P}^n_{\mathbb{Z}}$ onto the first factor is a smooth surjective morphism of relative dimension $(n-1) \cdot (r+1)$.*

Proof. We first show that f is flat. Note that f is a projective morphism and since the fibers at each point $q \in \mathbb{P}^n_{\mathbb{Z}}$ are isomorphic to $(\mathbb{P}^{n-1}_{k_q})^{r+1}$ it follows that

$$\sum_m (-1)^m \, \mathrm{h}^m(L_q, (\mathcal{O}_L)_q) = \sum_m (-1)^m h^m((\mathbb{P}^{n-1}_{k_q})^{r+1}, \mathcal{O}_{(\mathbb{P}^{n-1}_{k_q})^{r+1}}) = 1$$

where the last equality follows from for instance applying the Künneth formula and the standard calculations of the cohomology of the structure sheaf on projective n-space. Flatness now follows from [Vak13, Ex.24.7.A (d)]. Since the morphism f is finitely presented and flat and the fibers $(\mathbb{P}^{n-1}_{k_q})^{r+1}$ are smooth varieties of dimension $(n-1) \cdot (r+1)$ the morphism f is smooth of relative dimension $(n-1) \cdot (r+1)$. $\square$

Now let $F(-,-)$ be either of the sd-h-sheaves $\mathrm{Cycl}(-,-), \mathrm{Cycl}^{eff}(-,-)_{UI}$. Since f is flat of relative dimension $(n-1)\cdot(r+1)$ (Lemma 6.3.1) we get a morphism (subsection 2.5)

$$f^* : F(\mathbb{P}^n_{\mathbb{Z}}/\operatorname{Spec}(\mathbb{Z}), r) \to F(L/\operatorname{Spec}(\mathbb{Z}), (n-1)\cdot(r+1)+r).$$

Further letting $p : L \to G$ denote the composition

$$L \lhook\joinrel\longrightarrow \mathbb{P}^n_{\mathbb{Z}} \times G \longrightarrow G$$

we have that p is a proper morphism and thus by Corollary 2.5.6 we have a morphism

$$p_* F(L/\operatorname{Spec}(\mathbb{Z}), (r+1)n-1) \to F(G/\operatorname{Spec}(\mathbb{Z}), (r+1)n-1)$$

Thus we have a morphism of presheaves

$$Chow := p_* f^* : \mathrm{Cycl}^{eff}(\mathbb{P}^n_{\mathbb{Z}}/\operatorname{Spec}(\mathbb{Z}), r)_{UI} \to \mathrm{Cycl}^{eff}(G/\operatorname{Spec}(\mathbb{Z}), (r+1)n-1)_{UI}$$

which we call the *Chow homomorphism*. From now on we fix a Noetherian scheme S which shall be our base scheme and we shall denote $G \times_{\operatorname{Spec}(Z)} S/S$ by G/S , furthermore by abuse of notation we will denote the restriction of $Chow$ to the category of Noetherian S-schemes by

$$Chow := p_* f^* : \mathrm{Cycl}^{eff}(\mathbb{P}^n_S/S, r)_{UI} \to \mathrm{Cycl}^{eff}(G/S, (r+1)n-1)_{UI}.$$

The following lemma is stated without proof in [SV00].

Lemma 6.3.2 ([SV00, Lemma 4.4.12])**.** *For a positive integer d let $D = (d, \ldots, d)$ denote the $r+1$-tuple with every entry equal to d. The homomorphism $Chow$ takes the subsheaf $\mathrm{Cycl}^{eff}_d(\mathbb{P}^n_S/S, r)_{UI}$ to the subsheaf $\mathrm{Cycl}^{eff}_D(G/S, (r+1)n-1)_{UI}$.*

Proof. By Lemma 2.5.7 and Proposition 2.5.5 it is enough to show that for a morphism $\operatorname{Spec}(k) \to S$ with k an algebraically closed field the function $Chow(\operatorname{Spec}(k))$ takes an algebraic cycle of degree d in $\mathbb{P}^n_k$ to an algebraic cycle of multi-degree D in $(G/S) \times_S \operatorname{Spec}(k)$. To this extent note that since both proper push-forward and flat-pullback are linear and respect rational equivalence it is enough to prove that some r-dimensional hyperplane[2] in $\mathbb{P}^n_k$ is taken to an algebraic cycle of multi-degree $(1, 1, \ldots, 1)$. To this extent consider the r-dimensional linear subspace $W := V(x_1, \ldots, x_{n-r})$. Note that since all

[2]Here we are using the fact that $\mathrm{CH}(\mathbb{P}^n_k, r) = \mathbb{Z}[W]$ where W is any linear subspace of dimension r in $\mathbb{P}^n_k$ ([Ful98, Exampe 1.9.3])

fibers of the morphism $(f \times \mathrm{Spec}(k))$ are isomorphic to $(\mathbb{P}^{n-1}_k)^{\times r+1}$ it follows from [Vak13, Ex.11.4.C] that the scheme

$$f^*(W) := W \underset{\mathbb{P}^n_k}{\times} (L \underset{S}{\times} \mathrm{Spec}(k)) \cong \cap_{j=1}^{r+1} V(x_0 y_{0,j} + \sum_{i=n-r+1}^{n} x_i y_{i,j}) \subset \mathbb{P}^r_k \underset{k}{\times} (G/S \times \mathrm{Spec}(k))$$

is irreducible. Furthermore we have that

$$(p \underset{S}{\times} \mathrm{Spec}(k))(f^*(W)) \underset{\mathbb{P}^n_k}{\times} (L \underset{S}{\times} \mathrm{Spec}(k)) = V\left(\det(y_{l,j})_{l \in \{0\} \cup \{n-r+1 \le x \le n\}, 1 \le j \le r+1}\right).$$

By for instance picking appropriate affine charts and using Cramer's rule from linear algebra we see that the induced map of function fields $k(p(f^*(W))) \to k(f^*(W))$ is an isomorphism.

To conclude we have that

$$Chow(\mathrm{Spec}(k))(\mathrm{cycl}(W)) = \mathrm{cycl}\left(V\left(\det(y_{l,j})_{l \in \{0\} \cup \{n-r+1 \le x \le n\}, 1 \le j \le r+1}\right)\right)$$

and since the multi-homogeneous polynomial $\det(y_{l,j})_{l \in \{0\} \cup \{n-r+1 \le x \le n\}, 1 \le j \le r+1}$ has multi-degree $(1, 1, \ldots, 1)$ it follows from Example 6.1.10 that

$$\mathrm{cycl}\left(V\left(\det(y_{l,j})_{l \in \{0\} \cup \{n-r+1 \le x \le n\}, 1 \le j \le r+1}\right)\right)$$

has multi-degree $(1, \ldots, 1)$ completing the proof. □

We denote the restriction of $Chow$ to $\mathrm{Cycl}^{eff}_d(\mathbb{P}^n_S/S, r)_{UI}$ by $Chow_d$ and we restrict its target to $\mathrm{Cycl}^{eff}_D(G/S, (r+1)n-1)_{UI}$ which we can do by Lemma 6.3.2.

The following well known lemma is useful in the study of $Chow$ and $Chow_d$.

Lemma 6.3.3. *Let k be an infinite perfect field and Z a closed irreducible subvariety of $\mathbb{P}^n_k$. Suppose that $\dim Z = r$. Then for any point $x \in \mathbb{P}^n_k \setminus Z$ we can find $r+1$ hyperplanes of $\mathbb{P}^n_k$ such that their intersection contains x but does not intersect Z.*

Proof. This follows from Bertini's Theorem (see Thm. 12.4.2, Ex. 12.4.2A and Ex. 12.4.2B (b) in [Vak13]). □

Corollary 6.3.4 ([SV00, Lemma 4.4.12])**.** *The homomorphism Chow is a monomorphism.*

Proof. One easily reduces to proving that if k is an algebraically closed field then $Chow(\mathrm{Spec}(k))$ is an injection. This follows almost immediately from Lemma 6.3.3. □

Kollár gives a little more information about the Chow homomorphism. For a relative cycle $\mathcal{Z} = \sum a_i z_i \in \mathrm{Cycl}^{eff}(\mathbb{P}^n_S/S, r)(S)$ let Z_i be the closure of the points z_i in $\mathbb{P}^n_S$ and set $Ch(Z_i) := p(f^{-1}(Z_i)) \subset G/S$ where we give this the reduced induced subscheme structure. The following lemma is essentially the statement of [Kol96, Ch. I, Main Lemma 3.23.1.2].

Lemma 6.3.5. *The cycle $Chow(\mathcal{Z}) \in \mathrm{Cycl}^{eff}(G/S, (r+1)n-1)_{UI}(S)$ is of the form*

$$\sum a_i k_i \,\mathrm{cycl}_{G/S}(Ch(Z_i))$$

where the numbers k_i are powers of prime numbers occuring as the characteristics of the residue fields of generic points of S. If every generic fiber of $\mathrm{Supp}(\mathcal{Z}) \to S$ is geometrically reduced, then the numbers k_i are all equal to 1.

Proof. We will first prove the result in the simple case where $S = \mathrm{Spec}(k)$ for an algebraically closed field k. For a given i set $Z = Z_i$ and consider the morphism

$$\pi := p|_{f^{-1}(Z)} : f^{-1}(Z) \to p(f^{-1}(Z))_{\mathrm{red}}. \tag{6.3.1}$$

Since this is a proper morphism it follows easily from Theorem 1.1.12 that the set of points $H \in p(f^{-1}(Z))_{\mathrm{red}}$ such that the fiber $\pi^{-1}(H)$ is either empty or 0-dimensional forms an open subset of $p(f^{-1}(Z))_{\mathrm{red}}$. Hence there is an open subset over which the morphism π is finite and by Corollary 6.3.4 this open subset is necessarily dense in $p(f^{-1}(Z))_{\mathrm{red}}$. Let now x be a closed point in the r-dimensional scheme Z. By Bertini's Theorem (See [Vak13, Thm. 12.4.2]) we can find $r+1$ hypersurfaces $H_1, \dots, H_{r+1}$ such that their scheme theoretic intersection $\cap_{j=1}^{r+1} H_j$ is exactly the reduced point x. This shows that $\pi^{-1}(H_1, \dots, H_{r+1}) = (x, H_1, \dots, H_{r+1}) \in f^{-1}(Z)$. Hence by upper semicontinuity of rank/(degree of a finite morphism) we conclude that $k_i = 1$.

The general case can now readily be deduced from $Chow$ being a morphism of presheaves together with the following results: Lemma 2.3.13, Lemma 1.7.2 and Proposition 1.7.7. □

Corollary 6.3.6. *For a relative cycle $\mathcal{Z} \in \mathrm{Cycl}^{eff}(\mathbb{P}^n_S/S, r)(S)$ the morphism*

$$p|_{f^{-1}(\mathrm{supp}(\mathcal{Z}))} : f^{-1}(\mathrm{supp}(\mathcal{Z})) \to p(f^{-1}(\mathrm{supp}(\mathcal{Z}))) \tag{6.3.2}$$

is universally injective over a dense open subset of $p(f^{-1}(\mathrm{supp}(\mathcal{Z})))$

Proof. The set of points U of $p(f^{-1}(\mathrm{supp}(\mathcal{Z})))$ whose fibers are universally injective is by [GD67, (9.2.6)] constructible. Furthermore by Lemma 6.3.5 all the generic fibers are universally injective hence U is a constructible set of a Noetherian scheme containing all its generic points; thus it must necessarily be open. □

Chow with respect to a closed embedding

For a closed embedding $i : X \to \mathbb{P}^n_S$ over S denote by $\mathrm{Cycl}^{\mathit{eff}}_d((X,i)/S, r)_{UI}$ the presheaf that is the pullback in the category of presheaves of the following diagram

$$\begin{array}{ccc}
\mathrm{Cycl}^{\mathit{eff}}_d((X,i)/S, r)_{UI} & \longrightarrow & \mathrm{Cycl}^{eff}_d(\mathbb{P}^n_S/S, r)_{UI} \\
\downarrow & & \downarrow \\
\mathrm{Cycl}^{eff}(X/S, r)_{UI} & \xrightarrow{i_*} & \mathrm{Cycl}^{eff}(\mathbb{P}^n_S/S, r)_{UI}
\end{array}$$

where the lower horizontal arrow is the proper-pushforward induced by i_*. Note that $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ is both a sub-presheaf of $\mathrm{Cycl}^{eff}(X/S,r)_{UI}$ and $\mathrm{Cycl}_d^{eff}(\mathbb{P}_S^n/S,r)_{UI}$. The restriction of $Chow$ (resp. $Chow_d$) to $\mathrm{Cycl}^{eff}(X/S)$ (resp. $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$) will be denoted by $Chow(i)$ (resp. $Chow_d(i)$).

The Chow homomorphism over a field

Note that if we restrict ourselves to only working with reduced Noetherian schemes then cycl induces a morphism $\mathrm{H}_{\underline{d},\underline{n}} \to \mathrm{Cycl}_D^{eff}(G \times S/S,(r+1)n-1)_{UI}$ and moreover if we restrict ourselves even further to those Noetherian schemes which are normal then by Proposition 6.2.11 we can factorize $Chow_d$ through $\mathrm{H}_{\underline{d},\underline{n}}$. We will now try to understand what the image of this morphism is when we are working over $\mathrm{Spec}(k)$ for a reasonable field k. The following general lemma will be useful.

Lemma 6.3.7. *Consider the following commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ Z & \longrightarrow & S \end{array} \tag{6.3.3}$$

where p_1 is universally open. Let U be an open subset of the underlying topological space of Y and set $W := Z \setminus p_1(p_2^{-1}(U))$ where we consider this as a closed subset of the underlying topological space of Z. Then if $t : S' \to S$ is any morphism and we let $q : t^*Z \to Z$ be the induced projection then we have the following equality of sets

$$q^{-1}(W) = t^*Z \setminus (t^*p_1)((t^*p_2)^{-1}(t^*(U))) \tag{6.3.4}$$

Proof. It is enough to show that we have the following equality of sets

$$q^{-1}(p_1(p_2^{-1}(U))) = (t^*p_1)((t^*p_2)^{-1}(t^*(U))). \tag{6.3.5}$$

The inclusion " $\supset$ " follows easily from a diagram chase, and the inclusion " $\subset$ " follows easily from the universal property of fiber products and a diagram chase. $\square$

For a hypersurface $H \in \mathrm{Div}_{G/k}(\mathrm{Spec}(k))$ set

$$Z(H) := \{x \in \mathbb{P}_k^n \ : \ f^{-1}(\{x\}) \subset p^{-1}(H)\} \tag{6.3.6}$$

We will call this the *attempted Chow inverse of* H. Note that we have $Z(H) = \mathbb{P}_k^n \setminus f(p^{-1}(G/k \setminus H))$ and since f is a universally open morphism the set $Z(H)$ is necessarily a closed subset of $\mathbb{P}_k^n$. The following proposition is essentially [Kol96, Ch. I, Prop. 3.24.4]

Proposition 6.3.8. *For a hypersurface $H \in \mathrm{Div}_{G/k}(\mathrm{Spec}(k))$ we have*

1. $\dim Z(H) \leq r$.

2. *If H is irreducible and $\dim Z(H) = r$ then the closed set $Z(H)$ is necessarily irreducible. Furthermore if the field k is also perfect then there is some $d \in \mathbb{N}$ such that $Chow(Z(H)) = Chow_d(Z(H)) = H$.*

Proof. **For (1):** If $\dim Z(H) \geq r{+}1$ then the same is true for $\dim Z(H) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ where $\overline{k}$ denotes an algebraic closure of k. By Lemma 6.3.7 it is thus enough to show that this cannot happen when $k = \overline{k}$. Indeed since every codimension $r+1$ linear space meets a variety of dimension $\geq r+1$ in $\mathbb{P}^n_k$ it then follows that $p(f^{-1}(Z(H))) \subset H$ contains every closed point of G/k and must thus coincide with G/k which is absurd.

For (2): Let $V \subset Z(H)$ be an irreducible component of dimension r and assume first that the field k is infinite and perfect. For any point $x \notin V$ we apply Lemma 6.3.3 to find $r+1$ hyperplanes of $\mathbb{P}^n_k$ such that their intersection contains x but does not intersect V. Thus $x \notin Z(p(f^{-1}(V))) = Z(H)$. For the general case note that if Z is a variety of dimension r of finite type over a field k then $Z' = \mathrm{Spec}\, k(t) \times_{\mathrm{Spec}(k)} Z$ remains irreducible of dimension r and moreover the same is true for the base change $Z'' = \mathrm{Spec}(k(t)^{Perf} \times_{\mathrm{Spec}(k(t))} Z' = \mathrm{Spec}(k(t)^{Perf}) \times_{\mathrm{Spec}(k)} Z$, hence we can apply Lemma 6.3.7 to reduce to the case of an infinite perfect field which we have already solved.

The final statement of (2) follows from Lemma 6.3.5. $\square$

Corollary 6.3.9 ([Kol96, Cor. I.3.24.5])**.** *Let $i : X \to \mathbb{P}^n_S$ be a closed embedding. Then for a morphism $\mathrm{Spec}(k) \to S$ with k a perfect field we have that $Chow_d(i)(\mathrm{Spec}(k))$ gives a bijection between r dimensional cycles of degree d on $X \times_S \mathrm{Spec}(k) \subset \mathbb{P}^n_k$ and the set*

$$\left\{ \begin{array}{l} H \in \mathrm{H}_{\underline{d},\underline{n}}(\mathrm{Spec}(k)) : \dim Z(H_j) = r, Z(H_j) \subset X \times_S \mathrm{Spec}(k) \\ \textit{for every irreducible component } H_j \textit{ of } H \end{array} \right\}$$

Proof. Indeed if $\mathrm{cycl}_{G/k}(H) = \sum_j a_j H_j$ where we have $\dim Z(H_j) = r$ with $Z(H_j) \subset X \times_S \mathrm{Spec}(k)$ for all j then set

$$\mathcal{Z} := \sum a_j \,\mathrm{cycl}_{X \times_S \mathrm{Spec}(k)}(Z(H_j)_{\mathrm{red}})$$

Then by Proposition 6.3.8 we have $Chow(\mathcal{Z}) = \mathrm{cycl}_{G/k}(H)$ and as the cycle $\mathcal{Z}$ must have some degree and its image has degree $\underline{d}$ it is clear that $\mathcal{Z}$ must have degree d. $\square$

Construction of the Chow scheme of degree d cycles

Notation 6.3.10. As in Observation 6.2.24 we let $\mathscr{X}_d$ denote the universal hypersurface on $\mathrm{H}_{\underline{d},\underline{n}}(\mathbb{H}_{\underline{d},\underline{n}})$ and for a point $H \in \mathbb{H}_{\underline{d},\underline{n}}$ let $H^*(\mathscr{X}_d) \in$

$\mathbb{H}_{\underline{d},\underline{n}}(\mathrm{Spec}(k(H))$ denote the pullback of $\mathscr{X}_d$ along the map $\mathrm{Spec}(k(H)) \to \mathbb{H}_{\underline{d},\underline{n}}$. Furthermore we let $h_d : \mathbb{H}_{\underline{n},\underline{r}} \to S$ denote the structural map to S and for a closed embedding $i : X \to \mathbb{P}^n_S$ we let $H^* h_d^* X$ denote the fiber product $\mathrm{Spec}(K(H)) \times_{\mathbb{H}_{\underline{d},\underline{n}}} h_d^* X$ which is a closed subscheme of $\mathbb{P}^n_{k(H)}$.

Fix a closed embedding $i : X \to \mathbb{P}^n_S$ and set as a preliminary step

$$C_{r,d}((X,i)/S) := \left\{ \begin{array}{l} H \in \mathbb{H}_{\underline{d},\underline{n}} : \dim Z(H_j) = d \text{ and } Z(H_j) \subset H^* h_d^* X \text{ for every} \\ \text{irreducible component } H_j \text{ of } H^*(\mathscr{X}_d) \end{array} \right\}$$

where we recall that $Z(-)$ denotes the attempted Chow inverse of a hypersurface. When $X = \mathbb{P}^n_S$ we shall instead simply write $C_{r,d}(\mathbb{P}^n_S/S)$.

We want to show that $C_{r,d}((X,i)/S)$ is a closed subset of $\mathbb{H}_{\underline{d},\underline{n}}$.

Remark 6.3.11. We find it hard to understand the reasons for $C_{r,d}(\mathbb{P}^n_S/S)$ being closed given in [Kol96, Ch. I, Claim 3.25.1]. In loc.cit. one lets $h : \mathbb{H}_{\underline{d},\underline{n}} \to S$ denote the morphism to S and $h^*f : h^*L \to h^*\mathbb{P}^n$ and $h^*p : h^*L \to h^*G$ denote the induced morphisms from the base change from S to $\mathbb{H}_{\underline{d},\underline{n}}$. Setting

$$Z(\mathscr{X}_d) := \{x \in h^*\mathbb{P}^n \mid (h^*f^{-1}(x) \subset (h^*p)^{-1}(\mathscr{X}_d)\}$$

and letting $\pi : Z(\mathscr{X}_d) \to h^*\mathbb{P}^n \to \mathbb{H}_{\underline{d},\underline{n}}$ denote the induced morphism it is then claimed that by definition $H \in C_{r,d}(\mathbb{P}^n_S/S)$ if and only if every irreducible component of the fiber $\pi^{-1}(H)$ has dimension d. The following Example due to J. van Zelm shows that this claim is not correct and perhaps we have misunderstood what was meant in loc.cit.

Example 6.3.12. Let us consider the case $S = \mathrm{Spec}(k)$ for an algebraically closed field k and $n = 2$ and $r = 1$ so $G = (((\mathbb{P}^2_k)^\vee)/k)^2$. Consider the sets of pairs of hyperplanes

$$F_1 = \{(H_1, H_2) \in G \mid \exists t \in k \text{ such that } (t : 1-t : 0) \in H_1 \cap H_2 \subset \mathbb{P}^2_k\}$$

and

$$F_2 = \{(H_1, H_2) \in G \mid \exists t \in k \text{ such that } (t : 1-t : 0) \in H_1 \text{ and } (t : 0 : 1-t) \in H_2\}$$

Both F_1 and F_2 define irreducible degree $(1,1)$ hypersurfaces in G and their union F is a degree $(2,2)$ hypersurface in G. Now by construction we have

$$Z(F_1) = \{(t : 1-t : 0) \mid t \in k\} \subset \mathbb{P}^2_k \tag{6.3.7}$$

$$Z(F_2) = \{(1 : 0 : 0)\} \subset \mathbb{P}^2_k \tag{6.3.8}$$

$$Z(F) = Z(F_1 \cup F_2) = \{(t : 1-t : 0) \mid t \in k\} = Z(F_1). \tag{6.3.9}$$

This last equality shows that the fiber of π at the point of $\mathbb{H}_{\underline{2},\underline{2}}$ corresponding to F is exactly the set $Z(F)$ which is irreducible of dimension 1, while $Z(F_2)$ is just a point giving the contradiction.

Furthermore in the proof of this same claim it is claimed that upper-semicontinuity of fiber dimension tells us that the set where all irreducible components of the fibers are greater than a certain dimension is a closed subset. We do not know of any such theorem unless the morphism is both proper and flat in which case one can apply [GD67, (12.2.1)].

We will prove that $C_{r,d}((X,i)/S)$ is closed in several steps. The main ideas are extracted from [SV00, Sec. 4.4].

Notation 6.3.13. Let $\mathscr{X}_d^{irr} \subset {\mathbb{H}^{irr}}_{\underline{d},\underline{n}} \times_S G$ denote the universal irreducible hypersurface and let $h_d^{irr} : {\mathbb{H}^{irr}}_{\underline{d},\underline{n}} \to S$ denote its structural map to S. We now set

$$Z(\mathscr{X}_d^{irr}) := \{x \in {h_d^{irr}}^*\mathbb{P}^n_S \mid ({h_d^{irr}}^* f)^{-1}(x) \subset ({h_d^{irr}}^* p)^{-1}(\mathscr{X}_d^{irr})\} \tag{6.3.10}$$

and

$${C^{irr}}_{r,d}((X,i)/S) := \{H \in {\mathbb{H}^{irr}}_{\underline{d},\underline{n}} : \ \dim Z(H^*\mathscr{X}_d^{irr}) = r, \ Z(H^*\mathscr{X}_d^{irr}) \subset H^*{h_d^{irr}}^* X\}.$$

If $X = \mathbb{P}^n_S$ we shall instead denote this set by $C^{irr}_{r,d}(\mathbb{P}^n_S/S)$.

Lemma 6.3.14. *The set $C^{irr}_{r,d}(\mathbb{P}^n_S/S)$ is a closed subset of ${\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$.*

Proof. Since the fibers of $\mathscr{X}_d^{irr} \to {\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$ are all integral schemes it follows from Proposition 6.3.8 and Lemma 6.3.7 that if we let $\pi : Z(\mathscr{X}_d^{irr}) \subset {h_d^{irr}}^*\mathbb{P}^n_S \to {\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$ be the canonical map then we have

$$C^{irr}_{r,d}(\mathbb{P}^n_S/S) = \{H \in {\mathbb{H}^{irr}}_{\underline{d},\underline{n}} : \ \dim \pi^{-1}(H) \geq r\} \tag{6.3.11}$$

which by Theorem 1.1.12 is a closed subset. □

Corollary 6.3.15. *The set ${C^{irr}}_{r,d}((X,i)/S)$ is a closed subset of $C^{irr}_{r,d}(\mathbb{P}^n_S/S)$ hence also a closed subset of ${\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$.*

Proof. In the notation of the proof of Lemma 6.3.14 we recall that every fiber of the map

$$\pi : Z(\mathscr{X}_d^{irr}) \subset {h_d^{irr}}^*\mathbb{P}^n_S \to {\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$$

is irreducible and a fiber has dimension r if and only if it maps to a point of $C^{irr}_{r,d}(\mathbb{P}^n_S/S)$ hence $H \in {C^{irr}}_{r,d}((X,i)/S)$ if and only if the preimage of H under the morphism

$$Z(\mathscr{X}_d^{irr}) \cap {h_d^{irr}}^* X \to Z(\mathscr{X}_d^{irr}) \xrightarrow{\pi} {\mathbb{H}^{irr}}_{\underline{d},\underline{n}}$$

is an r-dimensional scheme. By Theorem 1.1.12 this is a closed subset. □

Lemma 6.3.16. *For $H \in \mathbb{H}_{\underline{d},\underline{n}}$ let $H' : \mathrm{Spec}(L) \to \mathbb{H}_{\underline{d},\underline{n}}$ denote the induced morphism from any field extension $L/k(H)$. The following are equivalent.*

1. $\dim Z(H_j) = r$ *and* $Z(H_j) \subset X \times_S \operatorname{Spec} k(H)$ *for every irreducible component* H_j *of* $H^* \mathscr{X}_d$.

2. $\dim Z(H'_i) = r$ *and* $Z(H'_i) \subset X \times_S \operatorname{Spec}(L)$ *for every irreducible component of* H'_i *of* $H'^* \mathscr{X}_d$.

Proof. Using Lemma 6.3.7 this is essentially just a simple diagram chase. □

Lemma 6.3.17. *The set* $C_{r,d}((X,i)/S)$ *is a constructible subset of* $\mathbb{H}_{\underline{d},\underline{n}}$.

Proof. By [Stacks, Tag 054J] it is enough to show that $C_{r,d}((X,i)/S)$ is the image of a morphism $f : Y \to \mathbb{H}_{\underline{d},\underline{n}}$ of finite presentation from a Noetherian scheme Y. Let $d_1, \ldots, d_k$ be natural numbers such that $d = d_1 + \ldots + d_k$. As we saw in Section 6.2 addition of relative effective Cartier divisors give us morphisms

$$\mathrm{H}^{irr}{}_{\underline{d_1},\underline{n}} \times \ldots \times \mathrm{H}^{irr}{}_{\underline{d_k},\underline{n}} \to \mathrm{H}_{\underline{d},\underline{n}}$$

Hence we get a morphism

$$q_{d_1,\ldots,d_k} : C^{irr}{}_{r,d_1}((X,i)/S) \times_S \ldots \times_S C^{irr}{}_{r,d_k}((X,i)/S) \to \mathbb{H}_{\underline{d},\underline{n}} \tag{6.3.12}$$

One easily checks that this morphism corresponds to the following hypersurface:

$$q^*_{d_1,\ldots,d_k} \mathscr{X}_d = \sum_{v=1}^{k} pr_v^*(j_v^* \mathscr{X}_{d_i}), \tag{6.3.13}$$

where $j_v : C^{irr}{}_{r,d_v}((X,i)/S) \to \mathbb{H}_{\underline{d_v},\underline{n}}$ is the obvious locally closed embedding and $pr_v : \prod_{v=1}^k C^{irr}{}_{r,d_v}((X,i)/S) \to C^{irr}{}_{r,d_v}((X,i)/S)$ denotes the projection. From this we see that if $t : \operatorname{Spec}(L) \to \prod_{v=1}^k C^{irr}{}_{r,d_v}((X,i)/S)$ is any morphism where L is a field then

$$t^* q^*_{d_1,\ldots,d_k} \mathscr{X}_d = \sum_{v=1}^{k} t^* pr_v^*(j_v^* \mathscr{X}_{d_v}) \tag{6.3.14}$$

where the hypersurfaces $t^* pr_v^*(j_v^* \mathscr{X}_{d_v})$ are irreducible and satisfy

$$\dim Z(t^* pr_v^*(j_v^* \mathscr{X}_{d_v})) = r$$

for all v. Hence by Lemma 6.3.16 it follows that $q_{d_1,\ldots,d_k} \circ t$ must factor through $C_{r,d}((X,i)/S)$.

Suppose now that $H \in C_{r,d}((X,i)/S)$ and for an algebraic closure $\overline{k(H)}$ of the residue field $k(H)$ let $\overline{H} : \operatorname{Spec}(\overline{k(H)}) \to \mathbb{H}_{\underline{d},\underline{n}}$ be the induced map. By Lemma 6.3.16 we have that if $\operatorname{cycl}(\overline{H}^* \mathscr{X}_d) = \sum_{u=1}^l m_u \operatorname{cycl}(H_u)$ where H_u are the irreducible components of $\overline{H}^* \mathscr{X}_d$ then we have $\dim Z(H_u) = r$ for every u and each of these have some multidegree of the form $\underline{d_u} = (d_u, \ldots, d_u)$ by Proposition 6.3.8. Moreover we have

$$\overline{H}^* \mathscr{X}_d = \sum m_u H_u$$

where the sum is the sum of effective Cartier divisors. One now readily checks that H is in the image of the map

$$\prod_u ({C^{irr}}_{r,d_u}((X,i)/S))^{m_u} \to \mathbb{H}_{\underline{d},\underline{n}}$$

thus proving that $C_{r,d}((X,i)/S)$ is exactly the set theoretic image of the map

$$\coprod_{k=1}^{d} \coprod_{\substack{(d_1,\ldots,d_k) \\ \sum_{v=1}^{k} d_v = d}} \prod_{j=1}^{k} {C^{irr}}_{r,d_j}((X,i)/S) \to \mathbb{H}_{\underline{d},\underline{n}}$$

which is constructible. $\square$

For later use we restate here the more precise statement that we just proved.

Observation 6.3.18. The set $C_{r,d}((X,i)/S) \subset \mathbb{H}_{\underline{d},\underline{n}}$ is exactly the set theoretic image of the map

$$\coprod_{k=1}^{d} \coprod_{\substack{(d_1,\ldots,d_k) \\ \sum_{v=1}^{k} d_v = d}} \prod_{j=1}^{k} {C^{irr}}_{r,d_j}((X,i)/S) \to \mathbb{H}_{\underline{d},\underline{n}}$$

The following technical, but highly important Lemma was inspired by the proof of [SV00, Lemma 4.4.4].

Lemma 6.3.19. *Let S be a Noetherian scheme and $X \to S$ be a morphism of finite type. Let T be an integral Noetherian scheme over S and let $\eta \in T$ denote the generic point of T. Suppose that E/k_η is a field extension and $\mathcal{Z} \in \mathrm{Cycl}(X_E, r)$. Then there is a proper h-covering $W \to T$ from an integral scheme W with field of functions $k(W) \subset E$ such that $k(W)$ is a finitely generated field extension of k_η and a relative cycle $\mathcal{Z}_W \in \mathrm{Cycl}(X \times_S W/W, r)_{UI}$ such that the pullback of $\mathcal{Z}_W$ to $\mathrm{Spec}(E)$ is the cycle $\mathcal{Z}$. If $\mathcal{Z} \in \mathrm{Cycl}^{eff}(X_E, r)$ then the relative cycle $\mathcal{Z}_W$ can be taken to be in $\mathrm{Cycl}^{eff}(X \times_S W/W, r)_{UI}$.*

Proof. Suppose that

$$\mathcal{Z} = \sum a_i z_i \in \mathrm{Cycl}(X_E, r).$$

Let Z_i denote the closure of z_i in X_E. By [GD67, Prop. (4.8.13)] there is a finitely generated field extension k_i of k_η and integral subscheme Z_i' of X_{k_i} such that $(Z_i') \times_{\mathrm{Spec}(k_i)} \mathrm{Spec}(E) = Z_i$ for each i. Let L be the composite of the field extensions k_i. For each i set $(Z_i'') := (Z_i') \times_{\mathrm{Spec}(k_i)} \mathrm{Spec}(L)$. Note that since we have a surjection $Z_i \to (Z_i)''$ from an integral scheme it follows that $(Z_i)''$ is irreducible and since the base change of (Z_i'') to $\mathrm{Spec}(E)$ is Z_i which is reduced it follows easily that $(Z_i)''$ must also be reduced hence $(Z_i)''$ is an integral scheme. Set $\mathcal{Z}'' := \sum a_i z_i'' \in \mathrm{Cycl}(X_L, r)$ where z_i'' are the generic points of Z_i'' and note that the flat pullback of $\mathcal{Z}''$ to X_E is exactly the cycle $\mathcal{Z}$. Let $t_1, \ldots, t_n$ be generators of the field extension L/k_η and let $\mathrm{Spec}(A) \to T$ be any open affine subset of T. Consider the composition

$$\varphi : A[T_1, \ldots, T_n] \to k_\eta[T_1, \ldots, T_n] \to L$$

where the last map is given by by $T_i \mapsto t_i$. Let $\mathfrak{p}$ denote the kernel of φ which clearly is a prime ideal. Then $V := \mathrm{Spec}(A[T_1, \ldots, T_n]/\mathfrak{p})$ is an integral scheme with function field L and V is a locally closed subscheme of $\mathbb{A}^n_T$ thus V is a locally closed subscheme of $\mathbb{P}^n_T$ and we let $\overline{V}$ denote the scheme theoretic image of V in $\mathbb{P}^n_T$. Then $\overline{V} \to T$ is a proper surjective morphism and $k(\overline{V}) = L$ hence we can consider the morphism $X_L \to \overline{V} \times_T X$. Let w_i denote the images of z_i'' in $\overline{V} \times_T X$ and W_i the closure of the points w_i (with the induced reduced subscheme structure). By Theorem 1.2.3 there is a blowup $W \to \overline{V}$ such that the strict transforms $\tilde{W}_i \to W$ are flat. Note that $W \to \overline{V} \to T$ is an h-covering of T. Furthermore by Lemma 2.3.4 it follows that $\tilde{W}_i \to W$ are universally equidimensional of dimension r and it is clear that the cycle

$$\mathcal{Z}_W := \sum a_i \,\mathrm{cycl}(\tilde{W}_i)$$

is contained in $\mathrm{Cycl}(W \times_S X/W, r)_{UI}$. Furthermore by Lemma 2.3.19 it is clear that the pullback of $\mathcal{Z}_W$ to $\mathrm{Spec}(E)$ is the cycle $\mathcal{Z}$ which completes the proof. □

Proposition 6.3.20. *The set $C_{r,d}((X,i)/S)$ is a closed subset of $\mathbb{H}_{\underline{d},\underline{n}}$.*

Proof. Since we have already proved that this is a constructible subset of $\mathbb{H}_{\underline{d},\underline{n}}$ (Lemma 6.3.17) it is enough to show that this subset is stable under specialization ([Stacks, Tag 0542]). To this extent suppose $H \in C_{r,d}((X,i)/S)$ and $H' \in \overline{\{H\}}$ is a specialization of H. By Lemma 6.3.16 and Corollary 6.3.9 we can find a cycle $\mathcal{Z} \in \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}(\mathrm{Spec}(k(H)^{Perf}))$ such that $Chow_d(i)(\mathcal{Z}) = \mathrm{cycl}((\mathscr{X}_d)_{k(H)^{Perf}})$, where $(\mathscr{X}_d)_{k(H)^{Perf}}$ denotes the base change of $\mathscr{X}_d$ along $\mathrm{Spec}(k(H)^{Perf}) \to \mathrm{Spec}(k(H)) \to \mathbb{H}_{\underline{d},\underline{n}}$. Now letting $T = \overline{\{H\}} \subset \mathbb{H}_{\underline{d},\underline{n}}$ be the closure of H with the reduced induced subscheme structure we have by Lemma 6.3.19 a proper surjective morphism $p : W \to T$ and a relative effective cycle $\mathcal{Z}_W \in \mathrm{Cycl}^{eff}(X \times_S W, r)$ where $k(H)^{Perf} \to T$ has a lifting to W such that the pullback of $\mathcal{Z}_W$ to $k(H)^{Perf}$ is the cycle $\mathcal{Z}$, hence since W is integral thus connected we must have $\mathcal{Z}_W \in \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}(W)$. Furthermore it is also easy to see that we must have $Chow_d(i)(\mathcal{Z}_W) = \mathrm{cycl}(\mathscr{X}_d \times_S W)$ and since every point of T has a lifting to W we then conclude by Lemma 6.3.16 that every point of T is contained in $C_{r,d}((X,i)/S)$ completing the proof. □

Endow the closed subset $C_{r,d}((X,i)/S)$ with the reduced induced subscheme structure. We shall here call this scheme the *Chow scheme of degree d relative cycles of dimension r with respect to i*. The reason for this name will become more apparent in the next sub-section, although the reader should be warned that there may be other schemes in the literature also referred to as "Chow schemes" which might not be isomorphic to the one provided here. We will get back to this issue later on.

Representability in the proper topology

Recall that for a Noetherian base scheme S and closed embedding $i : X \to \mathbb{P}^n_S$ we have that if $t : T \to S$ is a morphism from a reduced scheme such that the cycle $\mathrm{cycl}_{G\times_S T/T}(t^*\mathscr{X}_d)$ is in the image of $Chow_d(i)$ then the morphism t must necessarily factor through $C_{r,d}((X,i)/S)$. Ideally we would want the morphism $Chow_d(i)$ to factor through the representable presheaf $h_{C_{r,d}((X,i)/S)}$, however as a cycle of the form $Chow_d(i)(\mathcal{Z})$ is not necessarily induced by a relative effective Cartier divisor, this is likely too much to hope for. On the other hand Corollary 6.2.14 tells us that given any relative cycle $\mathcal{Z} \in \mathrm{Cycl}^{\mathit{eff}}_d((X,i)/S,r)_{UI}(T)$ there is a proper surjective morphism $p : T' \to T$ such that $Chow_d(i)(\mathrm{cycl}(p)(\mathcal{Z}))$ corresponds to a relative effective Cartier divisor on $G \times_S T'/T'$. This gives rise to the idea that if we sheafify our presheaves with respect to a suitable Grothendieck topology then our desired factorization becomes possible. We now make this precise:

Lemma 6.3.21. *Let $\mathcal{C}$ be a category and t a Grothendieck topology on $\mathcal{C}$ finer than a Grothendieck topology t_1 where the coverings of t_1 are either empty or singletons (i.e. they are of the form $\{X' \to X\}$). Let $\psi : F \to G$ be a map of presheaves and $H \subset G$ be a subpresheaf. Suppose that for any $X \in \mathcal{C}$ and $f \in F(X)$ there is some t-covering $p : X' \to X$ such that*

$$\psi(X')(p^*(f)) = p^*(\psi(X)(f)) \in H(X'),$$

Then the morphism $\psi_{\mathrm{t}} : F_{\mathrm{t}} \to G_{\mathrm{t}}$ factors through the monomorphism $H_{\mathrm{t}} \to G_{\mathrm{t}}$ thus inducing a map

$$\phi : F \to H_{\mathrm{t}}$$

(or equivalently $F_{\mathrm{t}} \to H_{\mathrm{t}}$) such that if $f \in F(X)$ is such that $\psi(X)(f) \in H(X)$ then $\phi(X)(f) = \psi(X)(f)^a$.

Proof. It is enough to prove the lemma in the case $\mathrm{t} = t_1$. For $f \in F(X)$ let $p : X' \to X$ be some t-covering such that $g' = p^*(\psi(X)(f)) \in H(X')$. It is clear that $pr_1^*(g') = pr_2^*(g') \in H(X' \times_X X')$ thus there is a unique element $\phi_p(f) \in H_{\mathrm{t}}(X)$ such that

$$p^*(\phi_p(f)) = (p^*(\psi(X)(f)))^a.$$

It is easy to see that if $q : X'' \to X$ is any other t-covering then we must necessarily have $\phi_p(f) = \phi_q(f)$; thus for $f \in F(X)$ we define $\phi(f) \in H_{\mathrm{t}}(X)$ to be this common element. We need to show that this gives a natural transformation. For a morphism $g : Y \to X$ and element $f \in F(X)$ pick some t-covering $p : X' \to X$ such that $p^*(\psi(X)(f)) \in H(X')$. Note that the pullback of g along p induces a t-covering $\{pr_Y : X' \times_X Y \to Y\}$ and $pr_Y^*(\psi(Y)(g^*(f))) \in H(X' \times_X Y)$. A diagram chase now shows that $g^*(\phi(f))$ satisfies the defining property of $\phi(g^*(f))$. $\square$

Definition 6.3.22. The *proper* topology on the category of Noetherian schemes over S is the topology where coverings are either empty or singletons of the form $\{p : X' \to X\}$ with p a proper surjective morphism. For a presheaf $\mathcal{F}$ on the category of Noetherian S-schemes we denote the sheafification of $\mathcal{F}$ with respect to the proper topology by $\mathcal{F}_{prop}$, if $\mathcal{F}$ is a representable presheaf $\mathcal{F} = h_{X/S}$ we shall instead denote the sheafification by $\mathrm{L}_{prop}(X/S)$.

Corollary 6.3.23. *The morphism*

$$Chow_d(i) : \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI} \to \mathrm{Cycl}_D^{eff}(G \times S/S, (r+1)n-1)_{UI}$$

induces a morphism of presheaves on the category of Noetherian S-schemes

$$\Phi_d : \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI} \to \mathrm{L}_{prop}(C_{r,d}((X,i)/S))$$

satisfying the following properties:

1. *If T is a normal Noetherian scheme over S then the image of $\Phi_d(T)$ is contained in the image of the canonical map $h_{C_{r,d}((X,i)/S)}(T) \to \mathrm{L}_{prop}(C_{r,d}((X,i)/S))(T)$.*
2. *For any Noetherian scheme T over S and relative cycle $\mathcal{Z} \in \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}(T)$ there is a proper surjective morphism $p : T' \to T$ such that $\Phi_d(T')(\mathrm{cycl}(p)(\mathcal{Z}))$ is in the image of the map $h_{C_{r,d}((X,i)/S)}(T') \to \mathrm{L}_{prop}(C_{r,d}((X,i)/S))(T')$.*
3. *If $\mathrm{Spec}(k) \to S$ is a morphism from a perfect field k then $\Phi_d(\mathrm{Spec}(k))$ is a bijection of sets.*

Moreover all these statements remain true after replacing the proper topology with the h-topology.

Proof. First apply Lemma 6.3.21 with ψ being the map

$$Chow_d(i) \circ (-)_{\mathrm{red}} : \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI} \circ (-)_{\mathrm{red}} \to \mathrm{Cycl}_D^{eff}(G \times S/S, (r+1)n-1)_{UI} \circ (-)_{\mathrm{red}}$$

and H the subpresheaf $h_{C_{r,d}((X,i)/S)} \circ (-)_{\mathrm{red}}$. Then from this and Lemma 6.2.15 we get the morphism

$$\Phi_d : \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI} \to C_{r,d}((X,i)/S)_{prop}.$$

The claimed properties of this map follow from Proposition 6.2.11, Corollary 6.2.14, Corollary 6.3.9 and Theorem 4.3.9. $\square$

Lemma 6.3.24. *Let X be a scheme of finite type over a Noetherian scheme S and T be any Noetherian scheme over S. Suppose that for each generic point $\tau \in T$ we are given a field extension $k(\tau) \subset E_\tau$ and $\mathcal{Z}_\tau \in \mathrm{Cycl}^{eff}(X_{E_\tau}, r)$. Then there is a proper surjective morphism $p : T' \to T$ and a relative cycle $\mathcal{Z}_{T'} \in \mathrm{Cycl}^{eff}(X/S, r)_{UI}(T')$ such that the following statements hold true:*

1. *The connected components of T' coincide with its irreducible components and p induces a bijection between generic points of T' and generic points of T.*

2. *For any generic point η of T' lying over $\tau \in T$ we have a tower of fields*

$$k(\tau) = k(p(\eta)) \subset k(\eta) \subset E_\tau$$

where $k(\eta)$ is a finitely generated field extension of $k(\tau)$. In particular for each τ the morphism $\mathrm{Spec}(E_\tau) \to T$ factors through T' via a map

$$e_\tau : \mathrm{Spec}(E_\tau) \to T'$$

3. *We have that $e_\tau^*(\mathcal{Z}_{T'}) = \mathcal{Z}_\tau$ for every generic point τ of T.*

Proof. Apply Lemma 6.3.19 to each of the irreducible components of T to obtain proper morphisms $p_i : T_i' \to T$ and cycles $\mathcal{Z}_{T_i'}$ pulling back to $\mathcal{Z}_{\tau_i}$. Set $T' := \coprod T_i'$ and let p be the map induced by the p_i. It is clear that p satisfies the desired properties and since $\mathrm{Cycl}^{eff}(X/S, r)_{UI}$ is a sheaf in the sd-h-topology it follows that the $\mathcal{Z}_{T_i'}$ glue to give a relative cycle $\mathcal{Z}_{T'}$ satisfying the desired properties. □

Theorem 6.3.25 (See also [SV00, Thm. 4.4.11], [SV00, Cor. 4.4.13])**.** *Let S be a Noetherian scheme and $i : X \to \mathbb{P}^n_S$ a closed embedding. Then the presheaf of relative cycles of degree d and dimension r $\mathrm{Cycl}^{\mathit{eff}}_d((X, i)/S, r)_{UI}$ is representable in the proper topology by the scheme $C_{r,d}((X, i)/S)$.*

Proof. It is enough to prove that the morphism

$$\Phi_d : \mathrm{Cycl}^{\mathit{eff}}_d((X, i)/S, r)_{UI} \to \mathrm{L}_{prop}(C_{r,d}((X, i)/S))$$

from Corollary 6.3.23 is proper-locally an isomorphism. Let T be any S-scheme with generic points $\tau_1, \ldots, \tau_l$. Setting $L_j = k(\tau_j)^{Perf}$ for $j = 1, \ldots, l$ we get induces canonical maps $\mathrm{Spec}(L_j) \to T$ for $j = 1, \ldots, l$. Consider the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Cycl}^{\mathit{eff}}_d((X, i)/S, r)_{UI}(T) & \xrightarrow{\Phi_d(T)} & \mathrm{L}_{prop}(C_{r,d}((X, i)/S))(T) \\
\downarrow & & \downarrow \\
\prod_{j=1}^{l} \mathrm{Cycl}^{\mathit{eff}}_d((X, i)/S, r)_{UI}(\mathrm{Spec}(L_j)) & \longrightarrow & \prod_{j=1}^{l} \mathrm{L}_{prop}(C_{r,d}((X, u)/S))(\mathrm{Spec}(L_j))
\end{array}$$

where the lower horizontal map is the map $\prod_{j=1}^{l} \Phi_d(\mathrm{Spec}(L_j))$. The leftmost vertical map is clearly injective and the lower horizontal map is a bijection by Corollary 6.3.23. Hence the upper horizontal map is injective proving that Φ_d is a local monomorphism with respect to the proper topology. To see that it is also a local epimorphism with respect to the proper topology let $g \in \mathrm{L}_{prop}(C_{r,d}((X,i)/S))$ be any element and let g' be its image in $\mathrm{L}_{prop}(C_{r,d}((X,u)/S))(\mathrm{Spec}(L_j))$. Since the lower horizontal map is a bijection there is a unique element $(\mathcal{Z})_{j=1}^{l} \in \prod_{j=1}^{l} \mathrm{Cycl}_r^{\mathit{eff}}((X,i)/S,r)_{UI}(\mathrm{Spec}(L_j))$ mapping to g'. By Lemma 6.3.24 we can find a proper surjective morphism $p : T' \to T$ such that the maps $\mathrm{Spec}(L_j) \to T$ all factor through p and a relative cycle $\mathcal{Z}_{T'} \in \mathrm{Cycl}^{eff}(X/S,r)(T')$ whose image in $\prod_{j=1}^{l} \mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}(\mathrm{Spec}(L_j))$ is $\mathcal{Z}$ (note that the leftmost vertical arrow in our diagram factors through $\mathrm{Cycl}^{eff}(X/S,r)(T')$. We now claim that $\Phi_d(T')(\mathcal{Z}_{T'}) = p^*(g)$. To prove this it is enough to show that the map

$$\mathrm{L}_{prop}(C_{r,d}((X,i)/S))(T') \to \prod_{j=1}^{l} \mathrm{L}_{prop}(C_{r,d}((X,i)/S))(\mathrm{Spec}(L_j))$$

is an injection. Using the description of

$$\mathrm{L}_{prop}(C_{r,d}((X,i)/S))(T')$$

given in Theorem 4.3.9 we have that if $g_1, g_2 : (T')^{awn} \to C_{r,d}((X,i)/S)$ are any two maps with the same image in $\prod_{j=1}^{l} \mathrm{L}_{prop}(C_{r,d}((X,i)/S))(\mathrm{Spec}(L_j))$ then we easily see that all generic points of $(T')^{awn}$ are necessarily contained in their equalizer which is a closed subscheme of $(T')^{awn}$. Since $(T')^{awn}$ is reduced it follows that $g_1 = g_2$. This completes the proof. □

Remark 6.3.26. A scheme such as $C_{r,d}((X,i)/S)$ that represents $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ in the proper topology is not unique up to isomorphism (see Corollary 4.3.11). However by Corollary 4.3.12 we know that the underlying topological spaces of any two schemes proper/h-representing $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ must necessarily be homeomorphic.

As an immediate consequence of Theorem 6.3.25 we get an alternative proof of the following theorem:

Corollary 6.3.27 ([Kol96, Ch. I, Thm. 3.21])**.** *Let S be a Noetherian scheme over a field of characeteristic zero. Then after restricting ourselves to the category of seminormal Noetherian schemes over S the presheaf $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ is representable by the semi-normalization of $C_{r,d}((X,i)/S)$.*

Proof. When S is of characteristic zero the sd-h and h-topology on Sch/S agree, hence by Lemma 6.1.15 the presheaf $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ is a sheaf in the h-topology. Furthermore in characteristic zero we have that $(-)^{sn}$ and $(-)^{awn}$ coincide and the desired result then follows from Theorem 6.3.25, Theorem 4.3.9 and Theorem 4.2.12. □

Applications to rational equivalence of algebraic cycles

We can extract information from our last few proofs to reinterpret rational equivalence in terms of rational curves on the Chow scheme. We will need the following lemma:

Lemma 6.3.28. *Let $f : X \to Y$ be a morphism of proper non-singular curves over a perfect field k. If the induced extension of function fields $k(Y) \subset k(X)$ is purely inseparable, then there is a unique isomorphism $Y = X$ such that f is the n-fold Frobenius of X/k.*

Proof. This is readily deduced from [Stacks, Tag 0CCZ]. □

Recall that if $i : X \to \mathbb{P}^n_S$ is a closed embedding then we have from Corollary 6.3.23 a morphism $\Phi_d : \mathrm{Cycl}^{\mathit{eff}}_d((X,i)/S,r)_{UI} \to \mathrm{L}_{prop}(C_{r,d}((X,i)/S))$ which is a bijection on k-points of S whenever k is a perfect field. The following proposition is a generalization of [Sam56, Thm.3].

Proposition 6.3.29. *Let k be a perfect field and $i : X \to \mathbb{P}^n_k$ a closed embedding. Then a cycle $\mathcal{Z} \in \mathrm{Cycl}(X,r)$ is rationally equivalent to zero if and only if there is a positive integer d and a rational curve $f : \mathbb{P}^1_k \to C_{r,d}((X,i)/\operatorname{Spec}(k))$ on $C_{r,d}((X,i)/\operatorname{Spec}(k))$ such that*

$$\mathcal{Z} = c(f(0)) - c(f(\infty)),$$

where $c(f(y))$ denotes the inverse image of the restriction of f to the k-point $y \in \mathbb{P}^1_k$ with respect to the map $\Phi_d(\operatorname{Spec}(k))$.

Proof. For necessity note that if $\mathcal{Z}$ is rationally equivalent to zero then by definition there is a relative r-cycle $\mathcal{W} \in \mathrm{Cycl}^{\mathit{eff}}_d((X,i)/\operatorname{Spec}(k),r)_{UI}(\mathbb{P}^1_k)$ such that

$$\mathcal{Z} = \mathrm{cycl}(t_0)(\mathcal{W}) - \mathrm{cycl}(t_\infty)(\mathcal{W}).$$

By Corollary 4.3.14 it follows that there is a finite purely inseparable field extension L of $k(t)$ such that if T denotes the normalization of $\mathbb{P}^1_k$ in L we have a morphism $f : T \to C_{r,d}((X,i)/\operatorname{Spec}(k))$ whose class represents $\Phi_d(\mathcal{W})$. By Lemma 6.3.28 it follows easily that f is a rational curve on $C_{r,d}((X,i)/\operatorname{Spec}(k))$ such that $c(f(0)) = \mathrm{cycl}(t_0)(\mathcal{W})$ and $c(f(\infty)) = \mathrm{cycl}(t_\infty)(\mathcal{W})$ which proves necessity.

For sufficiency suppose that $f : \mathbb{P}^1_k \to C_{r,d}((X,i)/\operatorname{Spec}(k))$ is a rational curve on $C_{r,d}((X,i)/\operatorname{Spec}(k))$. Let $t : \operatorname{Spec}(k(t)^{Perf}) \to \operatorname{Spec}(k(t)) \to \mathbb{P}^1_k$ be the obvious morphism and let $\mathcal{W}' \in \mathrm{Cycl}^{\mathit{eff}}_d((X,i)/\operatorname{Spec}(k),r)_{UI}(\operatorname{Spec}(k(t)^{Perf}))$ be the cycle that Φ_d maps to $f \circ t$. By Lemma 6.3.19 we have a surjective proper morphism $\pi : T \to \mathbb{P}^1_k$ from an integral scheme T with $k(t) \subset k(T) \subset k(t)^{Perf}$ and a relative cycle $\mathcal{W} \in \mathrm{Cycl}^{\mathit{eff}}_d((X,i)/\operatorname{Spec}(k),r)_{UI}(T)$ such that $\mathcal{W}$ pulls back to $\mathcal{W}'$. Furthermore the field extension $k(T)/k(t)$ is finitely generated thus (using Nullstellensatz) it is finite and purely inseparable. Furthermore after possibly normalizing T we can assume that it is a non-singular proper curve and moreover by Lemma 6.3.28 we may even assume that it is the projective line over k. It is clear that $\Phi_d(T)(\mathcal{W})$ coincides with the image of $f \circ \pi$ in $\mathrm{L}_{prop}(X/\operatorname{Spec}(k))(\mathbb{P}^1_k)$ and moreover the cycle $\mathrm{cycl}(t_0)(\mathcal{W})$ corresponds to $f \circ \pi \circ t_0 = f \circ t_0$ and similarly $\mathrm{cycl}(t_\infty)(\mathcal{W})$ corresponds to $f \circ t_\infty$ which proves sufficiency. $\square$

Remark 6.3.30. By essentially replacing the projective line with any smooth algebraic curve in Definition 2.5.16 we get the definition of *algebraic equivalence*. One can then prove an analogue of Proposition 6.3.29 for the notion of algebraic equivalence by considering all the connected components of the Chow scheme and then requiring the cycles involved to correspond to points lying in the same connected component of the Chow scheme.

6.4 The Chow monoid

The Chow monoid

Recall from Section 6.2 that for a Noetherian scheme S and a natural number r, addition of the equi-multi-degree hypersurfaces on $G = ((\mathbb{P}^n_S)^\vee)^{r+1}$ gives us maps

$$\beta^{d_1,d_2} : \mathbb{H}_{\underline{d_1},\underline{n}} \underset{S}{\times} \mathbb{H}_{\underline{d_2},\underline{n}} \to \mathbb{H}_{\underline{d_1+d_2},\underline{n}}$$

giving rise to the commutative graded monoid object

$$\mathbb{H}_{r,n} := \coprod_{d \geq 0} \mathbb{H}_{\underline{d},\underline{n}},\ \beta : \mathbb{H}_{r,n} \underset{S}{\times} \mathbb{H}_{r,n} \to \mathbb{H}_{r,n}.$$

Now for a given closed embedding $i : X \to \mathbb{P}^n_S$ and natural numbers $d_1, d_2 \in \mathbb{N}$ if we restrict the map β^{d_1,d_2} to the closed subscheme $C_{r,d_1}((X,i)/S) \times_S C_{r,d_2}((X,S)/S)$ then by Observation 6.3.18 this map must necessarily factor through $C_{r,(d_1+d_2)}((X,i)/S)$. Hence we have maps

$$\rho^{d_1,d_2} : C_{r,d_1}((X,i)/S) \underset{S}{\times} C_{r,d_2}((X,S)/S) \to C_{r,(d_1+d_2)}((X,i)/S) \tag{6.4.1}$$

giving rise to the commutative graded monoid object (Construction E.2.3).

$$C_r((X,i)/S) := \coprod_{d \geq 0} C_{r,d}((X,i)/S),\ \rho : C_r((X,i)/S) \underset{S}{\times} C_r((X,i)/S) \to C_r((X,i)/S) \tag{6.4.2}$$

Moreover since each graded piece of the scheme $C_r((X,i)/S)$ is a a closed subscheme of the respective graded piece of $\mathbb{H}_{r,n}$ we have that $C_r((X,i)/S)$ is a closed subscheme of $\mathbb{H}_{r,n}$ such that the following diagram commutes

$$\begin{tikzcd} C_r((X,i)/S) \times_S C_r((X,i)/S) \arrow[r, hook] \arrow[d, "\rho"] & \mathbb{H}_{r,n} \times_S \mathbb{H}_{r,n} \arrow[d, "\beta"] \\ C_r((X,i)/S) \arrow[r, hook] & \mathbb{H}_{r,n}. \end{tikzcd}$$

We shall call the (graded) commutative monoid object $C_r((X,i)/S)$ the *Chow monoid of r-dimensional relative cycles with respect to i.*

Remark 6.4.1. If the scheme S is Nagata then according to [Kol96, Ch. I, Thm. 4.13] the presheaf $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}$ considered as a presheaf on the category of semi-normal Noetherian Nagata schemes is coarsely represented by the semi-normalization of $C_{r,d}((X,i)/S)$ which we denote by $(C_{r,d}((X,i)/S))^{sn}$. Furthermore one easily sees that the canonical map

$$\coprod_{d\geq 0}\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI}\to \mathrm{Cycl}^{eff}(X/S,r)_{UI}$$

is Zariski-locally an isomorphism and so is the map

$$\coprod_{d\geq 0} h_{(C_{r,d}((X,i)/S))^{sn}}\to h_{\coprod_{d\geq 0}(C_{r,d}((X,i)/S)^{sn})}.$$

One can use the universal property of sheafification (with respect to the Zariski topology) to see that the scheme

$$\coprod_{d\geq 0}(C_{r,d}((X,i)/S)^{sn})$$

coarsely represents $\mathrm{Cycl}^{eff}(X/S,r)_{UI}$. This shows that the seminormalization of the Chow monoid $C_r((X,d)/S)$ is at least as a scheme independent of how X is embedded in $\mathbb{P}^n_S$.

h-representability in terms of Chow monoids

We will shortly show that if X is a scheme of finite type over the Noetherian scheme S then the presheaf $\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI}$ is after h-sheafification isomorphic to the h-sheafification of a locally Noetherian scheme provided that the morphism $X\to S$ factors as $X\xrightarrow{i}\mathbb{P}^n_S\to S$ where i is a closed embedding. Fix now such an embedding $i: X\to\mathbb{P}^n_S$ and note that addition of relative effective cycles gives us maps

$$\gamma^{d_1,d_2}:\mathrm{Cycl}_{d_1}^{\mathit{eff}}((X,i)/S,r)_{UI}\times\mathrm{Cycl}_{d_2}^{\mathit{eff}}((X,i)/S,r)_{UI}\to\mathrm{Cycl}_{(d_1+d_2)}^{\mathit{eff}}((X,i)/S,r)_{UI}$$

From Corollary 6.3.23 it follows that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Cycl}_{d_1}^{\mathit{eff}}((X,i)/S,r)_{UI}\times\mathrm{Cycl}_{d_2}^{\mathit{eff}}((X,i)/S,r)_{UI} \arrow[r, "\Phi_{d_1}\times\Phi_{d_2}"] \arrow[d, "\gamma^{d_1,d_2}"] & \mathrm{L}_h(C_{r,d_1}((X,i)/S))\times\mathrm{L}_h(C_{r,d_2}((X,i)/S)) \arrow[d] \\
\mathrm{Cycl}_{(d_1+d_2)}^{\mathit{eff}}((X,i)/S,r)_{UI} \arrow[r, "\Phi_{(d_1+d_2)}"] & \mathrm{L}_h(C_{r,(d_1+d_2)}((X,i)/S))
\end{tikzcd}$$

where the rightmost vertical arrow is the morphism induced from ρ^{d_1,d_2}. Applying sheafification and using Construction E.2.3 we obtain isomorphic monoid objects in the category of h-sheaves

$$C^{\bullet}((X,i)/S,r) := \coprod_{d\geq 0}(\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI})_h \cong \coprod_{d\geq 0}\mathrm{L}_h(C_{r,d}((X,i)/S)). \tag{6.4.3}$$

Furthermore the canonical inclusion maps $\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI} \to \mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI}$ induce morphisms of sheaves

$$\coprod_{d\geq 0}(\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI})_h \to (\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h \tag{6.4.4}$$

which one readily checks is in fact a morphism of commutative monoid objects in the category of h-sheaves on Noetherian schemes over S, in other words the following diagram commutes

$$\begin{tikzcd}
C^{\bullet}((X,i)/S,r)\times C^{\bullet}((X,i)/S,r) \arrow[r] \arrow[d] & (\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h \times (\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h \arrow[d] \\
C^{\bullet}((X,i)/S,r) \arrow[r] & (\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h.
\end{tikzcd}$$

Moreover we claim that the morphism given in (6.4.4) is an isomorphism. Indeed this map can be described as the following composition

$$\coprod_{d\geq 0}(\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI})_h \to (\coprod_{d\geq 0}\mathrm{Cycl}_d^{\mathit{eff}}((X,i)/S,r)_{UI})_h \to (\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h$$

where the first map is an isomorphism since sheafification commutes with coproducts and from Proposition 6.1.12 it follows easily that the second map is also an isomorphism. Finally we have a canonical map

$$\coprod_{d\geq 0}\mathrm{L}_h(C_{r,d}((X,i)/S)) \to \mathrm{L}_h(C_r((X,i)/S)) \tag{6.4.5}$$

which one readily checks is a morphism of commutative monoid objects and in fact an isomorphism since the morphism

$$\coprod_{d\geq 0} h_{C_{r,d}((X,i)/S)} \to h_{C_r((X,i)/S)}$$

is clearly Zariski-locally an isomorphism thus also locally an isomorphism with respect to the h topology. By combining the isomorphism of (6.4.3), (6.4.4) and (6.4.5) we obtain the following theorem:

Theorem 6.4.2. *Let S be a Noetherian scheme and $i: X \to \mathbb{P}^n_S$ be a closed embedding. Then the h-sheaves $(\mathrm{Cycl}^{\mathit{eff}}(X/S,r)_{UI})_h$ and $\mathrm{L}_h(C_r((X,i)/S))$ are isomorphic as h-sheaves of monoids.*

6.5 $\mathbb{Q}_+$-representability of relative cycles

In the previous section we saw that after sheafification in the h-topology we can (under reasonable assumptions) relate effective relative cycles to the sheaf represented by $C_r((X,i)/S)$. Using Theorem 5.0.1 this allows us to show that if we instead restrict ourselves to the category of seminormal Noetherian schemes over S and extend scalars, then we do not have to sheafify to understand relative cycles in terms of the Chow monoid.

Lemma 6.5.1. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then for any Grothendieck topology t coarser than the h-topology the induced map of presheaves*

$$(\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{UI})_t \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+} \tag{6.5.1}$$

is an isomorphism.

Proof. Since sheafification and localization preserves monomorphisms the induced map $(\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{UI})_h \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+}$ is a monomorphism. The fact that it is also an epimorphism follows easily from Proposition 2.3.27. □

Remark 6.5.2. Similarly if Λ is a sub semiring of $\mathbb{Q}_+$ containing the inverse of every element of $\exp.\mathrm{char}(S)$ the map

$$(\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{UI})_t \otimes_{\mathbb{N}} \Lambda \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{UI} \otimes_{\mathbb{N}} \Lambda$$

is an isomorphism.

Theorem 6.5.3. *Let S be a Noetherian scheme and $i : X \to \mathbb{P}^n_S$ a closed embedding and let $C_r((X,i)/S)$ denote the Chow monoid given in* (6.4.2). *Then after restricting the presheaves* $\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+}$ *and* $h_{C_r((X,i)/S)}$ *to the category of seminormal Noetherian schemes over S we have a natural transformation*

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+} \to h_{C_r((X,i)/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+$$

which is an isomorphism of presheaves of monoids.

Proof. By Lemma 6.5.1 and Theorem 6.4.2 we have an isomorphism

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,r)_{\mathbb{Q}_+} \cong \mathrm{L}_h(C_r((X,i)/S)) \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{6.5.2}$$

hence it is enough to prove that the natural map

$$h_{C_r((X,i)/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{L}_h(C_r((X,i)/S)) \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{6.5.3}$$

is an isomorphism. Letting $\mathbb{H}_{r,n}$ be the commutative monoid of (6.2.13) we have a commutative diagram

$$\begin{tikzcd}
h_{C_r((X,i)/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+ \arrow[r] \arrow[d] & \mathrm{L}_h(C_r((X,i)/S)) \otimes_{\mathbb{N}} \mathbb{Q}_+ \arrow[d] \\
h_{\mathbb{H}_{r,n}} \otimes_{\mathbb{N}} \mathbb{Q}_+ \arrow[r] & \mathrm{L}_h(\mathbb{H}_{r,n}) \otimes_{\mathbb{N}} \mathbb{Q}_+
\end{tikzcd}$$

As closed embeddings are monomorphisms in the category of schemes and the Yoneda embedding, the sheafification functor and localization of semi-modules all preserve monomorphisms it follows that the vertical arrows in our diagram are monomorphisms. Furthermore by Theorem 5.0.1 the lower horizontal morphism is an isomorphism and from the commutativity of the following diagram

$$\begin{tikzcd}
C_r((X,i)/S)/S \arrow[r, hook] \arrow[d, hook, "\Delta"] & \mathbb{H}_{r,n} \arrow[d, hook, "\Delta"] \\
(C_r((X,i)/S)/S)^d \arrow[r, hook] \arrow[d, "d\cdot\rho"] & (\mathbb{H}_{r,n})^d \arrow[d, "d\cdot\beta"] \\
C_r((X,i)/S) \arrow[r, hook] & \mathbb{H}_{r,n},
\end{tikzcd}$$

where d is any positive integer, we now easily deduce that the map given in (6.5.3) is an isomorphism. □

Remark 6.5.4. From Remark 5.3.1 and Remark 6.5.2 we see that Theorem 6.5.3 remains true after replacing $\mathbb{Q}_+$ with any subsemi-ring Λ containing the inverse of every element of $\mathrm{Exp.\,Char}(S)$.

Remark 6.5.5. In characteristic zero there are other presheaves of cycles which are representable. For example in the complex analytic setup Barlet defines in [Bar75] a presheaf F_X^n on the category of complex reduced spaces. Here X is a complex space and the functor F_X^n associates to each S the set of of analytic families of n-cycles of X parametrised by S ([KP96, Def.2.5]). It is proved in [Bar75] that the functor F_X^n is representable by a reduced complex space B_n. Moreover If X is a projective variety over the complex numbers, then the analytification of the classical Chow variety of n-cycles coincides with B_n ([KP96, Prop.2.8]).

More algebraically if S_0 is an affine scheme of characteristic zero and $X \to S$ is a smooth morphism. Then Angéniol defines in [Ang81] a presheaf C^p_{X/S_0} on the category of schemes over S_0 whose sections may be interpreted as families of cycles of codimension p. It is proved in op.cit. that this presheaf is representable by an algebraic space ([Ang81, Thm. 5.2.1]) which in the complex setup essentially coincides with Barlet's space after reduction (see [Ang81, Thm. 6.1.1] for the precise statement).

Note that it is not necessary to restrict oneself to the category of seminormal schemes for the presheaves of cycles due to Barlet and Angéniol to be representable. However this does not suggest that the seminormality assumption in Theorem 6.5.3 can be dropped. Indeed the presheaves F^n_X and C^p_{X/S_0} are not defined in terms of Definition 2.1.11, and their definitions are in fact arguably more complicated. Furthermore Theorem 3.2.9, Corollary 3.2.8, Theorem 4.3.9 and Example 7.2.3 all advocate the reduction to seminormal schemes. Restricting our presheaves to seminormal schemes has the advantage that we can use methods arising from the study of the h-topology, but it also has the disadvantage that there are interesting schemes such as the spectrum of the dual-numbers which becomes no different to the spectrum of a field in this setting.

6.6 An overview of the literature

In this chapter we provided a self-contained construction of the Chow schemes, which we did to a large extent by combining ideas from [SV00] and [Kol96]. The following table explains how several of the statements of this chapter compare to those found in the literature:

Comparison table			
Statement	Reference(s)	Statement comparison	Proof
Proposition 6.1.12	[SV00, Prop. 4.4.8]	Identical	Expands
Lemma 6.1.15	[SV00, p.78]	Extends	Added
Proposition 6.2.11	[SV00, Prop. 3.4.8]	Similar	Expands
Lemma 6.3.2	[SV00, Lemma 4.4.12]	Identical	Added
Lemma 6.3.5	[Kol96, Ch. I, M.Lem.3.23.1.2]	Similar	Expands
Proposition 6.3.8	[Kol96, Ch.I,Prop.3.24.4]	Similar	Expands
Proposition 6.3.20	[Kol96, Claim.3.25.1]	Identical	Different
Theorem 6.3.25	[SV00, Cor.4.4.13]	Similar	Incorporates similar ideas, but far from identical
Corollary 6.3.27	[Kol96, Ch. I, Thm. 3.21]	Similar	Different

Chapter 7

Relative zero cycles via symmetric powers

In this final chapter we prove the second main Theorem of the thesis. Just as in the higher dimensional case considered in the previous chapter this is done in two steps: first prove representability in a suitable Grothendieck topology then extend scalars and apply Theorem 5.0.1 to conclude. The first step can again be divided into two parts: first prove that after sheafification in the qfh topology the sheaf of effective relative zero cycles on X/S becomes isomorphic to the sheaf freely generated by X/S, and then prove that the latter sheaf is again isomorphic to $\mathrm{L}_{qfh}(\mathrm{Sym}^\bullet(X/S))$. Thus in the first section of the chapter we will deal with the first step where we follow [SV00] for the first needed isomorphism and [Voe96] for the second. Then in the second and final section we put everything together to obtain the final main Theorem. The final Proposition of the chapter tells us how our theorem relates to the use of symmetrization as considered in [SV96] and [Har16].

7.1 Freely generated representable sheaves

For an S-scheme X let $\mathbb{N}(X/S)$ denote the presheaf given by $T \mapsto \mathbb{N}(\mathrm{Hom}_S(T, X))$, where $\mathbb{N}(\mathrm{Hom}_S(T, X))$ is the free abelian monoid generated by the set $\mathrm{Hom}_S(T, X)$. We call this *the presheaf of abelian monoids freely generated by* h_X and denote its sheafification with respect to a Grothendieck topology t by $\mathbb{N}_t(X/S)$. In an analogous manner one can also define the presheaf of abelian groups freely generated by h_X which we denote by $\mathbb{Z}(X/S)$.

The following theorem is [SV00, Theorem 4.2.12]. The proof that we shall give is essentially the same as in loc. cit. the only difference being that we take a little more care when dealing with non-reduced schemes.

Theorem 7.1.1. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Then one has:*

1. *The sheaf* $(\mathrm{PropCycl}(X/S,0)_{UI})_{qfh}$ *is canonically isomorphic to the* qfh*-sheaf* $\mathbb{Z}_{qfh}(X/S)$ *of abelian groups freely generated by the sheaf of sets representable by* X*.*

2. *The sheaf* $(\mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI})_{qfh}$ *is canonically isomorphic to the* qfh*-sheaf* $\mathbb{N}_{qfh}(X/S)$ *of commutative monoids freely generated by the presheaf of sets represented by* X*.*

Proof. We will only prove the second statement. Note first that we have a canonical morphism $\mathbb{N}(X_{red}/S) \to \mathbb{N}(X/S)$ which is clearly qfh-locally an isomorphism. Consider now the closed embedding $X_{\mathrm{red}} \to X_{red} \times_S X$ such that the composition with the projection onto X_{red} gives identity and onto the second factor gives the canonical map $X_{\mathrm{red}} \to X$. One then has that $\delta := \mathrm{cycl}_{X_{\mathrm{red}} \times_S X}(X_{\mathrm{red}}) \in \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(X_{\mathrm{red}})$, which gives us a map $h_{X_{red}/S} \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}$, which again induces a unique morphism

$$\nabla : \mathbb{N}(X_{\mathrm{red}}/S) \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}. \tag{7.1.1}$$

We will show that this is a qfh-local isomorphism. Note that if $f : T \to X_{\mathrm{red}}$ is a morphism from a reduced scheme T then $\nabla(f) = \mathrm{cycl}_{T \times_S X}(\Gamma_f)$ where Γ_f denotes the graph of f. Since the irreducible components of a scheme form a qfh-covering one then easily sees that ∇ is qfh-locally a monomorphism. To show that it is qfh-locally an epimorphism it is enough to show that if T is an integral scheme over S and $\mathcal{Z} \in \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(T)$ then there is an qfh-covering $p : T' \to T$ such that $\mathrm{cycl}(p)(\mathcal{Z})$ is in the image of $\nabla(T')$. If $\mathcal{Z} = \sum a_i \,\mathrm{cycl}(Z_i)$ with Z_i integral schemes, then we will use induction on the number $N = \deg(\mathcal{Z}/T) := \sum a_i \deg(Z_i/T)$, where $\deg(Z_i/T)$ is the degree of the field extension of function fields. Note that if $N = 0$ there is nothing to prove. For the inductive step it is sufficient to show that if $p : Z_1 \to \mathrm{supp}(\mathcal{Z}) \to T$ is the map induced by $\mathcal{Z}$, then there exists a cycle $\mathcal{Z}_1 \in \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(Z_1)$ which is in the image of ∇ and such that

$$\deg(\mathrm{cycl}(p)(\mathcal{Z}) - \mathcal{Z}_1) < N.$$

Note that the cycle $\mathrm{cycl}(p)(\mathcal{Z})$ is of the form

$$\sum n_i \sum (m_j \,\mathrm{cycl}_{Z_1 \times_S X}(W_{i,j}))$$

where $W_{i,j}$ are the irreducible components of the schemes $Z_1 \times_T Z_i$. By Lemma A.2.1 and Example A.1.4 we easily see that

$$\sum m_j \deg(W_{i,j}/Z_1) = \deg(Z_i/T) \tag{7.1.2}$$

thus

$$\sum n_i m_j \deg(W_{i,j}/Z_1) = \deg(\mathcal{Z}/T). \tag{7.1.3}$$

Let $W_{1,1}$ be the irreducible component of $Z_1 \times_T Z_1$ which is the image of the diagonal embedding $Z_1 \to Z_1 \times_T Z_1$. Then $\mathrm{cycl}(W_{1,1}) \in \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(Z_1)$ hence we may set $\mathcal{Z}_1 = n_1 m_1 \,\mathrm{cycl}(W_{1,1})$. $\square$

Recall that we use $\mathrm{Sym}^\bullet(X/S)$ to denote the monoid object $\coprod_{n\geq 0}\mathrm{Sym}^n(X/S)$. The following Proposition is essentially [Voe96, Proposition.3.3.6]. The proof that we shall give is a little different from Voevodsky's, yet the central idea is the same.

Proposition 7.1.2. *Let $X \to S$ be a scheme of finite type over a Noetherian scheme S. Suppose in addition that X/S is flat and* AF. *Let*

$$\Psi : \mathbb{N}(X/S) \to h_{\mathrm{Sym}^\bullet(X/S)} \tag{7.1.4}$$

be the morphism from the presheaf of abelian monoids freely generated by h_X induced by

$$(f : T \to X) \mapsto i_1 \circ f \tag{7.1.5}$$

where $i_1 : X = \mathrm{Sym}^1(X/S) \to \mathrm{Sym}^\bullet(X/S)$ denotes the canonical inclusion. Then Ψ induces an isomorphism of the associated qfh-sheaves on the category of Noetherian S-schemes

$$(\Psi)_{qfh} : \mathbb{N}_{qfh}(X/S) \to \mathrm{L}_{qfh}(\mathrm{Sym}^\bullet(X/S)) \tag{7.1.6}$$

from the qfh-sheaf of commutative monoids freely generated by the presheaf of sets representable by X to the qfh-sheaf associated to the presheaf represented by $\mathrm{Sym}^\bullet(X/S)$.

Proof. We first produce an isomorphism in the other direction and prove that $(\Psi)_{qfh}$ must necessarily be its inverse.

For a given $d \in \mathbb{N}$ let $q_d : (X/S)^d \to \mathrm{Sym}^d(X/S)$ be the quotient map. Letting $pr_i : (X/S)^d \to X$ denote the projection to the i'th factor we have that the element $\sum_{i=1}^d pr_i \in \mathbb{N}(X/S)((X/S)^d)$ is obviously Σ_d invariant. Thus by Lemma 3.3.9 there exists a unique element $u_d \in \mathbb{N}_{qfh}(X/S)(\mathrm{Sym}^d(X/S))$ such that

$${q_d}^*(u_d) = (\sum_{i=1}^d pr_i)^a \in \mathbb{N}_{qfh}(X/S)((X/S)^d). \tag{7.1.7}$$

Now for a Noetherian scheme T over S and a morphism $f : T \to \mathrm{Sym}^\bullet(X/S)$, we let for each $d \in \mathbb{N}$, $f_d \in \mathrm{Hom}_S(f^{-1}(\mathrm{Sym}^d(X/S)), \mathrm{Sym}^d(X/S))$ be the map induced from restricting the morphism f. Note then that the sections

$$f_d^* u_d \in \mathbb{N}_{qfh}(X/S)(f^{-1}(\mathrm{Sym}^d(X/S)))$$

glue to give an element $\Theta(f) \in \mathbb{N}_{qfh}(X/S)(T)$. It is readily checked that this gives a natural transformation of presheaves of sets. We now claim that

$$\Theta : h_{\mathrm{Sym}^\bullet(X/S)} \to \mathbb{N}_{qfh}(X/S)$$

is in fact a natural transformation of presheaves of commutative monoids. For $f, g \in h_{\mathrm{Sym}^\bullet(X/S)}(T)$ check that $\Theta(T)(f) + \Theta(T)(g) = \Theta(T)(f + g)$. We may

clearly assume T to be connected and that there exist numbers $d, e \in \mathbb{N}$ such that
$$f^{-1}(\mathrm{Sym}^d(X/S)) = g^{-1}(\mathrm{Sym}^e(X/S)) = T.$$
We can then consider the pullback diagram

$$\begin{tikzcd}
T \times_{\mathrm{Sym}^d(X/S)\times_S \mathrm{Sym}^e(X/S)} (X/S)^d \times_S (X/S)^e \arrow[r, "p_{d,e}"] \arrow[d, "p_T"] & (X/S)^d \times_S (X/S)^e \arrow[d, "q_d \times_S q_e"] \\
T \arrow[r, "f_d \times_S g_e"] & \mathrm{Sym}^d(X/S) \times_S \mathrm{Sym}^e(X/S)
\end{tikzcd}$$

Letting
$$p_d : (X/S)^d \underset{S}{\times} (X/S)^e \to (X/S)^d\,;$$
$$p_e : (X/S)^d \underset{S}{\times} (X/S)^e \to (X/S)^e$$
denote the two projections and
$$i : (X/S)^d \underset{S}{\times} (X/S)^e \to (X/S)^{d+e}$$
the isomorphism given by Convention 1.6.15, we have the following equalities:
$$p_T^*(f_d^* u_d) = p_{d,e}^* p_d^* q_d^* u_d = p_{d,e}^* p_d^* (\sum_{i=1}^{d} pr_i)^a; \tag{7.1.8}$$
$$p_T^*(g_e^* u_e) = p_{d,e}^* p_e^* q_e^* u_e = p_{d,e}^* p_e^* (\sum_{j=1}^{e} pr_j)^a; \tag{7.1.9}$$
$$p_T^*(f_d + g_e)^* u_{d+e} = p_{d,e}^* i^* (\sum_{l=1}^{d+e} pr_l). \tag{7.1.10}$$
Hence we have
$$p_T^*(f_d^* u_d + g_e^* u_e) = p_T^*(f_d + g_e)^* u_{d+e} \tag{7.1.11}$$
thus since p_T is a qfh-covering we conclude that $\Theta(T)(f+g) = \Theta(T)(f) + \Theta(T)(g)$.

We now claim that Θ is qfh-locally an isomorphism. Note that for any morphism $f : T \to X$ we can compose with the open embedding $X \to \mathrm{Sym}^\bullet(X/S)$ to obtain a map $g : T \to \mathrm{Sym}^\bullet(X/S)$ such that $g_1 = f$, hence
$$\Theta(T)(g) = [f]^a \in \mathbb{N}_{qfh}(X/S)(T).$$
Since Θ is a homomorphism of presheaves of monoids we now easily conclude that Θ is necessarily a qfh-local epimorphism.

Suppose now that $f, g \in h_{\mathrm{Sym}^\bullet(X/S)}(T)$ satisfy

$$\Theta(T)(f) = \Theta(T)(g).$$

We want to prove that there exists a qfh-covering $p_i : T_i \to T$ such that $f \circ p_i = g \circ p_i$ for all i. Clearly we may again assume T to be connected and that there exist numbers $d, e \in \mathbb{N}$ such that

$$T = f^{-1}(\mathrm{Sym}^d(X/S)) = g^{-1}(\mathrm{Sym}^e(X/S)).$$

Consider the two following qfh-coverings:

$$f_d^* q_d : T \underset{\mathrm{Sym}^d(X/S)}{\times} (X/S)^d \to T; \tag{7.1.12}$$

$$g_e^* q_e : T \underset{\mathrm{Sym}^e(X/S)}{\times} (X/S)^e \to T. \tag{7.1.13}$$

We can find a common refinement of both these coverings say $\{\pi_\alpha : T_\alpha \to T\}_{\alpha \in A}$ with T-morphisms

$$r_\alpha : T_\alpha \to T \underset{\mathrm{Sym}^d(X/S)}{\times} (X/S)^d; \tag{7.1.14}$$

$$s_\alpha : T_\alpha \to T \underset{\mathrm{Sym}^e(X/S)}{\times} (X/S)^e. \tag{7.1.15}$$

Letting

$$p_d : T \underset{\mathrm{Sym}^d(X/S)}{\times} (X/S)^d \to (X/S)^d; \tag{7.1.16}$$

$$p_e : T \underset{\mathrm{Sym}^e(X/S)}{\times} (X/S)^e \to (X/S)^e \tag{7.1.17}$$

denote the projections, we may then suppose that we have the equality

$$\sum_{i=1}^{d} pr_i \circ p_d \circ r_\alpha = \sum_{j=1}^{e} pr_j \circ p_e \circ s_\alpha \tag{7.1.18}$$

for every $\alpha \in A$. From this we easily see that we must have $d = e$ and if we set $x = p_d \circ r_\alpha$ and $y = p_e \circ s_\alpha$ then by the universal property of fibre products, we easily see that there is some $\sigma \in \Sigma_d$ such that $x = \rho(\sigma) \circ y$, where $\rho(\sigma)$ is the automorphism of $(X/S)^d$ induced by σ. Hence we must have

$$q_d \circ x = q_d \circ y : T_\alpha \to \mathrm{Sym}^d(X/S) \tag{7.1.19}$$

from which we conclude that

$$f_d \circ \pi_\alpha = g_e \circ \pi_\alpha \tag{7.1.20}$$

for every $\alpha \in A$ which proves that Θ is also qfh-locally a monomorphism. This proves that the induced morphism

$$(\Theta)_{qfh} : \mathrm{L}_{qfh}(\mathrm{Sym}^{\bullet}(X/S)) \to \mathbb{N}_{qfh}(X/S) \tag{7.1.21}$$

is an isomorphism.

It now remains to prove that the composition $(\Psi)_{qfh} \circ (\Theta)_{qfh}$ is the identity as this implies that $(\Psi)_{qfh}$ must be the inverse of the isomorphism $(\Theta)_{qfh}$. To this extent note that if T is any Noetherian S-scheme with generic points $\eta_1, \ldots, \eta_r \in T$ and $k_1, \ldots, k_r$ are algebraic closures of the residue fields $k(\eta_1), \ldots, k(\eta_r)$ then the map

$$\mathrm{L}_{qfh}(\mathrm{Sym}^{\bullet}(X/S))(T) \to \prod_{i=1}^{r} \mathrm{L}_{qfh}(\mathrm{Sym}^{\bullet}(X/S))(\mathrm{Spec}(k_i))$$

is injective (this is easily deduced from Theorem 4.3.9). Hence it is enough to show that the composition $(\Psi)_{qfh} \circ (\Theta)_{qfh}$ coincides with the identity on the spectrum of an algebraically closed field. To this extent let k be an algebraically closed field and f be a morphism

$$f : \mathrm{Spec}(k) \to \mathrm{Sym}^{d}(X/S) \xrightarrow{i_d} \mathrm{Sym}^{\bullet}(X/S)$$

where $i_d : \mathrm{Sym}^{d}(X/S) \to \mathrm{Sym}^{\bullet}(X/S)$ is the canonical inclusion. Since the quotient map $q_d : (X/S)^d \to \mathrm{Sym}^{d}(X/S)$ is finite there is a map

$$f'_d : \mathrm{Spec}(k) \to (X/S)^d \tag{7.1.22}$$

such that we have

$$f = i_d \circ q_d \circ f'_d. \tag{7.1.23}$$

Thus by construction we have

$$(\Theta)_{qfh}(\mathrm{Spec}(k))(f) = \sum_{i=1}^{d} (pr_i \circ f'_d) \tag{7.1.24}$$

and from Remark 1.6.17 we see that

$$(\Psi)_{qfh}(\mathrm{Spec}(k))(\sum_{i=1}^{d} (pr_i \circ f'_d)) = f. \tag{7.1.25}$$

This completes the proof.

□

Reinterpreting rational equivalence of zero cycles

In Section 6.3 we gave a modern proof of the fact that rational equivalence of cycles can be understood by means of rational curves on the Chow scheme. We are now almost ready to state and prove the analogue for zero cycles and symmetric powers.

Lemma 7.1.3. *Let* $\mathrm{Spec}(k) \to S$ *be a morphism from a perfect field. Then the sheafification at* $\mathrm{Spec}(k)$

$$\mathrm{PropCycl}(X/S, r)_{UI}(\mathrm{Spec}(k)) \to (\mathrm{PropCycl}(X/S, r)_{UI})_{qfh}(\mathrm{Spec}(k))$$

is an isomorphism.

Proof. Recall that every quasi-finite morphism to the spectrum of a field is necessarily a finite morphism. Since every field extension of a perfect field is separable the claim follows now easily from Lemma 2.3.21. □

For a flat AF-scheme X/S we have by Theorem 7.1.1 and Proposition 7.1.2 an isomorphism

$$\Xi : (\mathrm{PropCycl}(X/S, 0)_{UI})_{qfh} \to \mathrm{L}_{qfh}(\mathrm{Sym}^{\bullet}(X/S))$$

where both $\mathrm{PropCycl}(X/S, 0)_{UI}$ and $h_{\mathrm{Sym}^{\bullet}(X/S)}$ take the same values as their sheafifications at k-points of S where k is a perfect field. The following Proposition is a generalization of [Ful98, Example 1.6.3].

Proposition 7.1.4. *Let k be a perfect field and $X \to \mathrm{Spec}(k)$ a finite type morphism such that X is* AF. *Then a zero cycle $\mathcal{Z} \in \mathrm{Cycl}(X, 0)$ is rationally equivalent to zero if and only if there is a rational curve $f : \mathbb{P}^1_k \to \mathrm{Sym}^{\bullet}(X/\mathrm{Spec}(k))$ such that*

$$\mathcal{Z} = c(f(0)) - c(f(\infty))$$

where $c(f(y))$ is the cycle on X corresponding to the k-point $f(y)$ on $\mathrm{Sym}^{\bullet}(X/\mathrm{Spec}(k))$ for $y \in \mathbb{P}^1_k(k)$.

Proof. This is proved mutatis mutandis as Proposition 6.3.29. □

Remark 7.1.5. Proposition 7.1.4 is inspired by V. Guletskiĭ's (unpublished) proof in the characteristic zero case.

7.2 Relative zero cycles in terms of the monoid of symmetric powers

Theorem 7.2.1. *Let $X \to S$ be a flat finite type morphism to a Noetherian scheme S such that X/S is* AF *(Definition 1.5.26). Then after restrcting the presheaves of monoids $h_{\mathrm{Sym}^{\bullet}(X/S)}$ and $\mathrm{PropCycl}^{\mathit{eff}}(X/S, 0)_{\mathbb{Q}_+}$ to the category* *of Noetherian seminormal schemes over S we get an isomorphism of presheaves of monoids:*

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{\mathbb{Q}_+} \to h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+ \tag{7.2.1}$$

Proof. By Lemma 6.5.1, Theorem 7.1.1 and Proposition 7.1.2 we always have the following isomorphisms

$$\mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{\mathbb{Q}_+} \cong \mathbb{N}_{qfh}(X/S) \otimes_{\mathbb{N}} \mathbb{Q}_+ \cong \mathrm{L}_{qfh}(\mathrm{Sym}^\bullet(X/S)) \otimes_{\mathbb{N}} \mathbb{Q}_+, \tag{7.2.2}$$

and after restricting ourselves to the category of Noetherian seminormal schemes over S we obtain by Proposition 1.6.19 and Theorem 5.0.1 an isomorphism of monoids

$$h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \mathbb{Q}_+ \to \mathrm{L}_{qfh}(\mathrm{Sym}^\bullet(X/S)) \otimes_{\mathbb{N}} \mathbb{Q}_+$$

which completes the proof. □

Remark 7.2.2. From Remark 5.3.1 and Remark 6.5.2 we see that Theorem 7.2.1 remains true after replacing $\mathbb{Q}_+$ with any subsemi-ring Λ containing the inverse of every element of $\mathrm{Exp.\,Char}(S)$.

Example 7.2.3. Let S be a Noetherian scheme of characteristic zero, and consider the maps

$$\nabla : \mathbb{N}(X_{\mathrm{red}}/S) \to \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}\,;$$
$$\Psi : \mathbb{N}(X/S) \to h_{\mathrm{Sym}^\bullet(X/S)}$$

From (7.1.1) and (7.1.4) respectively. These maps are qfh-locally isomorphisms hence by Corollary 3.2.8 and Theorem 4.3.9 it follows that if the scheme S is seminormal then we have an isomorphism

$$\Omega(S) : \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(S) \to h_{\mathrm{Sym}^\bullet(X/S)}(S). \tag{7.2.3}$$

Suppose in addition that the seminormal scheme S is integral. For a proper relative zero cycle $\mathcal{Z} = \sum_i a_i \,\mathrm{cycl}(Z_i) \in \mathrm{PropCycl}^{\mathit{eff}}(X/S,0)_{UI}(S)$ where Z_i are integral closed subschemes of X set $d = d(\mathcal{Z}) := \sum a_i[k(Z_i) : k(S)]$. We claim that the map $\Omega(S)(\mathcal{Z}) : S \to \mathrm{Sym}^\bullet(X/S)$ factors through the inclusion

$$i_d : \mathrm{Sym}^d(X/S) \to \mathrm{Sym}^\bullet(X/S).$$

Indeed it is enough to show that if K is an algebraic closure of $k(S)$ and $t : \mathrm{Spec}(K) \to S$ is the obvious map, then the map $\Omega(S)(\mathcal{Z}) \circ t$ factors through i_d. From Proposition 1.7.9 it follows that $d(\mathrm{cycl}(t)(\mathcal{Z})) = d$ hence we can reduce to the case where S is an algebraically closed field, but then the claim follows easily from the construction of the maps ∇ and Ψ. In particular if there exists a cycle $\mathcal{Z}$ such that $d(\mathcal{Z}) = 1$ then there exists a section of the map $X \to S$. The necessity of such a section strongly motivates the restriction to seminormal schemes in Theorem 7.2.1. Indeed consider for instance the cusp $C = V(y^2 - x^3) \subset \mathbb{A}^2_k$ where k is a field of characteristic 0. The seminormalization of this scheme is the normalization $X = \mathbb{A}^1_k \to C$, and the scheme C is geometrically unibranched. Thus the generic point $\eta \in X$ is an element of $\mathrm{PropCycl}^{\mathit{eff}}(X/C, 0)_{UI}(C)$, but the morphism $X \to C$ has no sections.

Proposition 7.2.4. *Let $X \to S$ be a flat finite type morphism to a Noetherian scheme S such that X/S is* AF. *Let Λ be any sub semi-ring of $\mathbb{Q}_+$ such* $\mathrm{Exp.\,Char}(S) \subset \Lambda^{\times}$[1] *Let*

$$\Upsilon : h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \Lambda \to \mathrm{PropCycl}^{\mathit{eff}}(X/S, 0)_{UI} \otimes_{\mathbb{N}} \Lambda$$

be the following compositions of isomorphisms (Theorem 7.2.1, Remark 7.2.2)

$$h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \Lambda \to \mathbb{N}_{qfh}(X/S) \otimes_{\mathbb{N}} \Lambda \to \mathrm{PropCycl}^{\mathit{eff}}(X/S, 0)_{UI} \otimes_{\mathbb{N}} \Lambda.$$

Then for any normal Noetherian scheme T over S the inverse of $\Upsilon(T)$ is given by symmetrization as in [Har16, Definition 3.7.1].

Proof. Let $f : \mathrm{Spec}(k) \to \mathrm{Sym}^d(X/S) \xrightarrow{i_d} \mathrm{Sym}^\bullet(X/S)$, where i_d denotes the inclusion $\mathrm{Sym}^d(X/S) \to \mathrm{Sym}^\bullet(X/S)$ be a morphism from an algebraically closed field k. From the construction of Υ we see that

$$\Upsilon(\mathrm{Spec}(k))(f) = \sum_{i=1}^{d} \mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(\Gamma_{pr_i \circ f'_d}), \tag{7.2.4}$$

where $f'_d : \mathrm{Spec}(k) \to (X/S)^d$ is any map such that we have $f_d = q_d \circ f'_d$, where q_d denotes the quotient $(X/S)^d \to \mathrm{Sym}^d(X/S)$. Setting $\alpha_i := \mathrm{cycl}_{\mathrm{Spec}(k) \times_S X}(\Gamma_{pr_i \circ f'_d})$, we easily see that the symmetrization of α_i which we denote by $sym(\alpha_i)$ is exactly the morphism $i_1 \circ pr_i \circ f'_d$. Thus we have

$$sym(\Upsilon(\mathrm{Spec}(k))(f))) = \sum_{i=1}^{d} sym(\alpha_i) = \sum_i (i_1 \circ pr_i \circ f'_d) = f. \tag{7.2.5}$$

This proves the Proposition in the special case where T is an algebraically closed field. For the general case note that if $\{\eta_i\}_{i=1}^n$ are the generic points of a normal Noetherian scheme T and we let k_i be algebraic closures of the residue fields $k(\eta_i)$ then we have a commutative diagram

$$\begin{tikzcd}
h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \Lambda(T) \arrow[r, "\Upsilon(T)"] \arrow[d] & \mathrm{PropCycl}^{\mathit{eff}}(X/S, 0)_{UI} \otimes_{\mathbb{N}} \Lambda(T) \arrow[d] \\
\prod_{i=1}^n h_{\mathrm{Sym}^\bullet(X/S)} \otimes_{\mathbb{N}} \Lambda(\mathrm{Spec}(k_i)) \arrow[r] & \prod \mathrm{PropCycl}^{\mathit{eff}}(X/S, 0)_{UI} \otimes_{\mathbb{N}} \Lambda(\mathrm{Spec}(k_i)).
\end{tikzcd}$$

[1] For a semi-ring A we let $A^\times$ denote the units of A.

where the lower horizontal map is $\prod \Upsilon(\mathrm{Spec}(k_i))$. Since both the vertical arrows are inclusions and symmetrization also commutes with pullbacks ([Har16, Proposition 3.6.6]) we conclude the proof. $\square$

Remark 7.2.5. Proposition 7.2.4 gives another reason why the functors from Theorems 3.7.5, 3.7.7 and 3.7.8 of [Har16] are full (see Remark 3.7.11 of op cit). Furthermore our proposition taken together with Proposition 2.5.13 also shows that Theorem 6.8 of [SV96] is a special case/restriction of Theorem 7.2.1.

Epilogue

The author would like to list some open questions related to the material we have seen in this thesis.

1. It is known that semi and weak normality are both stable under smooth base change. The case of weak normality can be deduced from Manaresi's description of the weak normalization (see [Man80]), while for seminormality the proof is rather different as it uses Traverso's gluings (see [GT80]). Is it possible to prove an analogous result for $(B/A)_\eta$ as defined in Section 4.1?

2. It was already asked in [AB69] if the seminormalization of a Noetherian scheme is necessarily Noetherian. As far as we are aware this question remains open to this day. If this could be answered affirmatively then the results of this thesis requiring seminormality would be somewhat strengthened as this would mean that seminormalization induces an endofunctor on the category of Noetherian schemes.

3. Are there interesting cases of morphisms $X \to S$ and dimension functions on S such that Suslin-Voevodsky's relative cycles on X/S of dimension r are exactly the cycles on X/S with δ-dimension r as defined in [Stacks, Tag 02QQ]?

4. If the answer to the previous question is affirmative, is it then possible to generalize Proposition 7.1.4 and/or Proposition 6.3.29 to the setting of rational equivalence given in [Stacks, Tag 02RW], [Stacks, Tag 02S3]?

5. Is it possible to say anything about the locus of points on S where a relative cycle on X/S becomes rationally equivalent to zero?

Appendix A

The length of a module

A.1 Definition and basic properties

Definition A.1.1. Let M be an A-module. A *chain* of submodules of a module M is a sequence $(M_i)_{0\leq i\leq n}$ of submodules of M such that

$$M = M_0 \supsetneq M_1 \supset \ldots \supsetneq M_n = 0$$

The *length* of the chain is n.

A *composition series* of M is a maximal chain, that is one in which no extra submodules can be inserted or equivalently that each quotient M_{i-1}/M_i has no submodules except 0 and itself which again is equivalent to $M_{i-1}/M_i \cong A/\mathfrak{m}_\mathfrak{i}$ where $\mathfrak{m}_i$ is some maximal ideal of A.

Proposition A.1.2 ([AM69, Proposition 6.7])**.** *Suppose that M has a composition series of length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.*

Definition A.1.3. If the A module M has a composition series of length n we say that the *length* of the A-module M is n and denote this by

$$\text{length}_A(M) = n$$

Example A.1.4 ([AM69, Prop.6.10])**.** For a finite dimensional k-vector space V we have $\text{length}_k(V) = \dim_k(V)$.

Example A.1.5. Let A be a DVR with uniformizer π and valuation v . For any nonzero $a \in A$, consider the A-module $M = A/(a)$. The sub A-modules of M are the ideals of $A/(a)$ which correspond to the ideals of A containing a. The ideals of A are ideals of the form (π^i), with $i \in \mathbb{N}$, and we have $(\pi^{\text{val}(a)}) = (a)$, hence

$$M = A/(a) \supsetneq (\pi)A/(a) \supsetneq \ldots \supsetneq (\pi^{\text{val}(a)-1})A/(a) \supsetneq (\pi^{\text{val}(a)})A/(a) = 0$$

is a composition series of M, thus $\text{length}_A(M) = \text{val}(a)$.

Proposition A.1.6 ([AM69, Proposition 6.8]). *The module M has a composition series if and only if M satisfies both the ascending and descending chain conditions.*

Corollary A.1.7. *Let A be a Noetherian one-dimensional ring. If $x \in A$ is not a zero divisor, then $A/(x)$ has finite length.*

Proof. By Krull's Hauptidealsatz we have $\dim A/(x) = 0$, thus A is Artinian and thus satisfies both chain conditions. $\square$

Proposition A.1.8 ([AM69, Prop. 6.9]). *The length* $\mathrm{length}_A(M)$ *is an additive function on the class of all A-modules of finite length.*

Corollary A.1.9. *Let A be a one dimensional Noetherian ring. If $a, b \in A$ where either a or b is not a zero divisor, then*

$$\mathrm{length}_A(A/(ab)) = \mathrm{length}_A(A/(a)) + \mathrm{length}_A(A/(b))$$

and these lengths are finite.

Proof. The last assertion follows from Corollary A.1.7. For the first apply Proposition A.1.8 to the exact sequence

$$0 \to A/(a) \xrightarrow{\cdot b} A/(ab) \to A/(b) \to 0$$

$\square$

Lemma A.1.10. *If M has finite length with a composition series*

$$M_0 = M \supsetneq M_1 \supsetneq \ldots \supsetneq M_n = 0.$$

Then

(1) *each $M_{i-1}/M_i \cong A/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*

(2) *given a maximal ideal $\mathfrak{m} \subset A$ we have*

$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = \mathrm{length}_{A_\mathfrak{m}}(M_\mathfrak{m}).$$

(3) *If $\mathfrak{p} \subset A$ is a prime ideal which is not maximal then* $\mathrm{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$. *Hence*

$$\mathrm{length}_A(M) = \sum_{\mathfrak{p}} \mathrm{length}_{A_\mathfrak{p}}(M_\mathfrak{p}),$$

where the sum is over all the prime ideals of A.

Proof. **For (1)**: since M_{i-1}/M_i is simple we have a maximal ideal $\mathfrak{m}_i$ of A such that $M_{i-1}/M_i \cong A/(\mathfrak{m})$. **For (2) (and (3))**: Suppose $\mathfrak{p} \subset A$ is any prime ideal. If we localize the composition series provided in the statement of this Lemma at $\mathfrak{p}$ we can start from $(M_0)_\mathfrak{p}$ and see if it is equal to $(M_1)_\mathfrak{p}$. If so, then we remove $(M_0)_\mathfrak{p}$ from the chain and move our way down the chain in this fashion removing $(M_{i-1})_\mathfrak{p}$ whenever it coincides with $(M_i)_\mathfrak{p}$. In this way we obtain a composition series of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Note that in the aforementioned process of making a composition series of $M_\mathfrak{p}$ we kept $(M_i)_\mathfrak{p}$ if and only if $(M_i/M_{i+1})_\mathfrak{p} \neq 0$ which is the case if and only if $\mathfrak{m}_i = \mathfrak{p}$. Hence the result follows and moreover we have also shown that if $\mathfrak{p}$ is not maximal then we must necessarily have $\text{length}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$. □

Lemma A.1.11. *Let A be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an A-module with $\mathfrak{m}M = 0$, then the length of M as an A-module agrees with the dimension of M as an $A/\mathfrak{m}$ vector space. The length is finite if and only if M is a finitely generated A-module.*

Proof. Under the assumptions on M we see that any sub A-module of M is also a sub $A/\mathfrak{m}$-module of M, and so any composition series of M as an A-module is also a composition series of M as an $A/\mathfrak{m}$-module, thus the first assumption follows. The second follows by picking a basis for M as an $A/\mathfrak{m}$-module and producing the obvious composition series. □

A.2 Length and homomorphisms

Lemma A.2.1 ([Stacks, Tag 02M0])**.** *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \dots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$\text{length}_A(M) = \sum\nolimits_{i=1,\dots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \, \text{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $\text{length}_A(M) < \infty$.

Proof. Let $M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_N = 0$ be a composition series for M as a B-module. We have that $M_k/M_{k+1} \cong B/\mathfrak{m}_{j(k)}$ with $j(k) \in \{1, \dots, n\}$. Set $d_i = \#\{k \in \{0, \dots, N\} \mid j(k) = i\}$, then by additivity of length we have that

$$\begin{aligned} \text{length}_A(M) &= \sum_{k=0}^{N-1} \text{length}_A(M_k/M_{k+1}) \\ &= \sum_{i=1}^{n} d_i \, \text{length}_A(B/\mathfrak{m}_i) = \sum_{i=1}^{n} \text{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}) \cdot [k(\mathfrak{m}_i) \ : \ k(\mathfrak{m})]. \end{aligned}$$

□

Lemma A.2.2. *Let $A \to B$ be a flat local homomorphism of Artinian local rings. Then*

$$\mathrm{length}_B(B) = \mathrm{length}_A(A) \cdot \mathrm{length}_B(B/\mathfrak{m}B)$$

where $\mathfrak{m}$ is the maximal ideal of A.

Proof. Let

$$A = I_0 \supsetneq I_1 \supsetneq \ldots \supsetneq I_r = 0$$

be a composition series of A as a module over itself. Then $I_i/I_{i+1} \cong A/\mathfrak{m}$ for each i and since B is flat over A we have

$$(I_iB/I_{i+1}B) \cong (I_i/I_{i+1}) \otimes_A B \cong B/\mathfrak{m}B$$

for each i. Thus we have a filtration

$$B = I_0B \supsetneq I_1B \supsetneq \ldots \supsetneq I_rB = 0$$

and using that length is additive, we then obtain

$$\mathrm{length}_B(B) = r \cdot \mathrm{length}_B(B/\mathfrak{m}B).$$

□

Lemma A.2.3 ([Stacks, Tag 02M1])**.** *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$\mathrm{length}_A(M)\,\mathrm{length}_B(B/\mathfrak{m}_AB) = \mathrm{length}_B(M \otimes_A B).$$

In particular, if $\mathrm{length}_B(B/\mathfrak{m}_AB) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\mathrm{length}_A(M) = \infty \Rightarrow \mathrm{length}_B(M \otimes_A B) = \infty$. Next, assume $\mathrm{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \mathrm{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_AB$. Thus the lemma follows. □

Appendix B

Commutative Semi-Algebra

B.1 Monoids, semi-rings and semi-modules

Definition B.1.1. A *monoid* is a pair $(A, \cdot)$ where A is a set and $\cdot$ a binary operation $A \times A \to A$ satisfying the following two properties:

(Associativity) for all $a, b, c \in A$ we have

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(Identity) there exists an element $e \in A$ such that $a \cdot e = e \cdot a = a$ for all $a \in A$.

A monoid $(A, \cdot)$ is said to be *commutative* or *abelian* if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.

Definition B.1.2. A *Semi-ring*[1] is an algebraic structure $(A, +, \cdot, 0, 1)$ satisfying the following properties:

1. $(A, +, 0)$ is A a *commutative monoid*.
2. $(A, \cdot, 1)$ is a commutative monoid.
3. $a(b + c) = ab + ac$ for all $a, b, c \in A$.
4. $a \cdot 0 = 0$ for all $a \in A$.

Example B.1.3. The prime example of a semi-ring, and especially in this thesis, is the natural numbers $\mathbb{N}$ together with the usual addition, multiplication, 0 and 1. Another example typical for us is the set of positive rational numbers denoted $\mathbb{Q}_+$.

Definition B.1.4. A *semi-ring homomorphism* from A to B is a function $f : A \to B$ satisfying the following properties:

[1] We will only be interested in commutative semi-rings in this text, and therefore omit the adjective "commutative".

1. $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in A$.

2. $f(0) = 0$ and $f(1) = 1$.

Definition B.1.5. Let A be a semi-ring. A *semi-module* over A, or A-semi-module for short, is a commutative monoid $(M, +, 0)$ together with a function $A \times M \to M$, called the *multiplication of scalar map*, satisfying the following conditions:

1. $a(m+n) = am + an$ for all $a \in A$ and $m, n \in M$.

2. $(a+b)m = am + bm$ for all $a, b \in A$ and $m \in M$.

3. $(ab)m = a(bm)$ for all $a, b \in A$ and $m \in M$.

4. $1m = m$ for all $m \in M$

5. $0m = 0$ for all $m \in M$ and $a0 = 0$ for all $a \in A$.

Definition B.1.6. A *homomorphism of semi-modules* $f : A \to B$ is a homomorphism of the underlying commutative monoids such that for all $a \in A$ and $m \in M$ we have $f(am) = af(m)$.

B.2 Localization

We introduce localization of semi-rings and semi-modules following [Nas18].

Definition B.2.1. A *multiplicatively closed* subset of a semi-ring A is a subset S containing the multiplicative identity which is closed under multiplication, in otherwords the action $\cdot$ endows S with the structure of a commutative monoid $(S, \cdot, 1)$.

Furthermore an element $a \in A$ is a *unit* , or *invertible* , if there exists a $b \in A$ such that $ab = 1$. Such an element b is clearly unique if it exists and we call it the *inverse* of a written $b = a^{-1}$. More generally if M is an A-semi-module then $a \in A$ is said to be a *unit* in M if the induced endomorphism of M given by $m \mapsto am$ is an isomorphism.

Definition B.2.2. Let S be a multiplicatively closed subset of a semi-ring A.

A *localization* of A with respect to S is a semi-ring homomorphism $\lambda_S : A \to S^{-1}A$ such that the image of any $s \in S$ in $S^{-1}A$ is a unit and if $\varphi : A \to B$ is any other semi-ring homomorphism satisfying this property then there exists a unique semi-ring homomorphism $\varphi' : S^{-1}A \to B$ such that $\varphi = \varphi' \circ \lambda_S$.

Similarly if M is a semi-module over a semi-ring A then a *localization* of M with respect to S is an A-semi-module homomorphism $\gamma_S : M \to S^{-1}M$ such that every $s \in S$ is a unit in $S^{-1}M$ and any other such map factors uniquely through γ_S.

Proposition B.2.3. *Let A be a semi-ring, S a multiplicatively closed subset of A and M an A-semi-module. Then the localization of A (resp. of M) with respect to S exists.*

Proof. We first consider the case of M. Let $\sim$ be the relation on $S \times M$ defined by $(s, m) \sim (t, n)$ if there exists some $u \in S$ such that $utm = usn$. Set $S^{-1}M := S \times M/\sim$ and denote the class of (s, m) by $m/s \in S^{-1}M$. Define addition by

$$m/s + n/t = (tm + sn)/st. \tag{B.2.1}$$

It is easy to check that this is well defined and gives $S^{-1}M$ the structure of an A-semi-module by $a(m/s) = am/s$. Furthermore it is straightforward to check that every $s \in S$ is a unit in $S^{-1}M$. Let $\gamma_S : M \to S^{-1}M$ be given by $m \mapsto m/1$. If $\varphi : M \to N$ is any homomorphism of A-semi-modules where every element of S is a unit in N then φ then letting $\varphi' : S^{-1}M \to N$ be given by taking m/s to the element n in N such that $sn = \varphi(m)$ is easily seen to be a well defined morphism and we clearly have $\varphi = \varphi' \circ \gamma_S$. Furthermore uniqueness of φ' is immediate hence we have proved the existence of the localization of M with respect to S.

The localization of the semi-ring A with respect to S is similar. Indeed consider first A as a module over itself and localize this module with respect to S. This gives us an A-module homomorphism $\gamma_S : A \to S^{-1}A$ given by $a \mapsto a/1$. Furthermore we can define multiplication on $S^{-1}A$ as usual:

$$(a/s) \cdot (b/t) = (ab/st). \tag{B.2.2}$$

It is a routine exercise to see that $(A, +, \cdot, 0, 1)$ is a semi-ring and that the A-module homomorphism γ_S is also a semi-ring homomorphism. Furthermore if $\varphi : A \to B$ is a semi-ring homomorphism taking every $s \in S$ to a unit in B then by considering B as an A-semi-module and from what we have shown the map $\varphi' : S^{-1}A \to B$ given by $a/s \mapsto \varphi(s)^{-1} \cdot \varphi(a)$ is the unique map which φ factors through and one checks easily that φ' is a semi-ring homomorphism thus completing the proof. $\square$

Throughout by $S^{-1}A, S^{-1}M$ will denote the semi-ring and semi-module constructed in the proof of Proposition B.2.3 respectively.

Note that the A-semi-module $S^{-1}M$ can be given the structure of an $S^{-1}A$-semi-module by setting

$$(a/s) \cdot (m/t) = (am/st), \tag{B.2.3}$$

and if we ever speak of $S^{-1}M$ as an $S^{-1}A$-semi-module this is the multiplication of scalar-map we are referring to.

Example B.2.4. Note that the positive rational numbers $\mathbb{Q}_+$ is the localization of the semi-ring $\mathbb{N}$ with respect to the multiplicatively closed subset $\mathbb{N} \setminus \{0\}$.

Notation B.2.5. For a number $n \in \mathbb{N}$ we denote by $\mathbb{N}[1/n]$ the localization of $\mathbb{N}$ with respect to the multiplicatively closed subset $\{n^i\}_{i\in\mathbb{N}}$.

It is easy to see from the universal property of localization that it is functorial. For a morphism of A-semi-modules $\varphi : M \to N$ we let $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ denote the canonically induced A-semi-module homomorphism. Furthermore it is clear that $S^{-1}\varphi$ can also be understood as an $S^{-1}A$-module homomorphism in a canonical manner.

Lemma B.2.6. *Let $\varphi : M \to N$ be a homomorphism of semi-modules over the semi-ring A and S a multiplicatively closed subset of A. The following assertions hold true:*

1. *If φ is injective then so is the induced map $S^{-1}\varphi : S^{-1}M \to S^{-1}N$.*

2. *If φ is an isomorphism (in the category of A-semi-modules) then so is the induced map $S^{-1}\varphi$.*

Proof. **For Item 1:** If we have $m_1/s_1, m_2/s_2 \in M$ mapping to the same element in $S^{-1}N$ then we necessarily have

$$\varphi(m_1)/s_1 = \varphi(m_2)/s_2 \tag{B.2.4}$$

hence there is some $t \in S$ such that

$$ts_2\varphi(m_1) = ts_1\varphi(m_2) \tag{B.2.5}$$

but then by injectivity of φ we obtain

$$t(s_2m_1) = t(s_1m_2) \tag{B.2.6}$$

hence $m_1/s_1 = m_2/s_2 \in S^{-1}M$.

For Item 2: It is easy to see that a map of A-semi-modules is an isomorphism if and only if the underlying set theoretic map is a bijection. From Item 1 it is enough to show that $S^{-1}\varphi$ is surjective, but this is obvious. □

Remark B.2.7. As mentioned in the proof of Lemma B.2.6 it is obvious that the Lemma is also true for surjective morphisms. Moreover we also have that $S^{-1}(-)$ takes monomorphisms (resp. isomorphisms) in the category of A-semi-modules to monomorphisms (resp. isomorphisms) in the category of $S^{-1}A$-semi-modules. Furthermore by considering $S^{-1}\varphi$ as a morphism of semi-modules over A it will be a monomorphism in this category if φ is.

B.3 Tensor product

Definition B.3.1. Let A be a semi-ring and M, N, P semi-modules over A.

A set theoretic map $f : M \times N \to P$ is said to be A-*bilinear* if for any $m \in M$ and $n \in N$ the two induced maps $N \to P, M \to P$ given by $x \mapsto f(m, x)$ and $y \mapsto f(y, n)$ respectively are both A-semi-module homomorphisms.

Definition B.3.2. Let A be a semi-ring and M, N semi-modules over A. A *tensor product*[2] of M and N over A is an A-bilinear map $M \times N \to M \otimes_A N$ such that for any A-bilinear map $M \times N \to P$ there exists a unique A-semi-module homomorphism $M \otimes_A N \to P$ making the following diagram commute

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] & P \\ M \otimes_A N \arrow[ur, dotted, "\exists!"] & \end{tikzcd} \tag{B.3.1}$$

Proposition B.3.3. *Let A be a semi-ring and M, N semi-modules over A. Then the tensor product $M \otimes_A N$ exists.*

Proof. We provide the construction given in for instance [Ban13]. Let F be the free commutative monoid generated by the set $M \times N$ and let $\sim$ be the congruence relation on F generated by all pairs of the form

$$((m+m', n), (m, n)+(m', n)), ((m, n+n'), (m, n)+(m, n')), ((am, n), (m, an)), \tag{B.3.2}$$

$m, m' \in M; n, n' \in N; a \in A$. Set $M \otimes_A N := F/\sim$ and denote the image of $m \times n$ in $M \otimes_A N$ by $m \otimes n$. The commutative monoid $M \otimes_A N$ is then an A-semi-module with

$$a(m \otimes n) = (am) \otimes n = m \otimes (an). \tag{B.3.3}$$

To see that the canonical map $M \times N \to M \otimes_A N$ satisfies the universal property we refer the reader to [Kat04]. $\square$

One sees immediately from the universal property of the tensor product that given any A-semi-module N then $(-) \otimes_A N$ gives an endofunctor on the category of A-semi-modules.

[2] As explained in [Ban13] there are two non-isomorphic tensor products in the litterature, both called the tensor product and both written $M \otimes_A N$. The notion defined here is the one analogous to the standard notion we know from commutative algebra. The tensor product found in for instance [Gol99] satisfies a different universal property.

B.4 Extension of scalars

Let M be an A-semi-module and $A \to B$ be a semi-ring homomorphism. Then using the universal property of the tensor product we see that the A-semi-module $M \otimes_A B$ can be given the structure of a B-semi-module given by

$$b' \cdot (m \otimes b) = m \otimes (b'b). \tag{B.4.1}$$

We say that this B-semi-module has been obtained from M by *extension of scalars*.

If N is a semi-module over a semi-ring B and we have semi-ring homomorphism $\varphi : A \to B$ then N inherits an A-semi-module structure given by $a \cdot n := \varphi(a) \cdot n$. We say then that this A-module was obtained from N by *restriction of scalars*. Just as in the case of modules over rings, extension and restriction of scalars of semi-modules over semi-rings gives an adjunction [Kat04, Proposition 4.1].

Just as in the case of modules over rings we have that extending scalars of a semi-module to a localization of the semi-ring is the same as localizing the semi-module:

Lemma B.4.1. *Let A be a semi-ring, S a multiplicatively closed subset of A and M a semi-module over A. The map*

$$S^{-1}A \otimes_A M \to S^{-1}M \tag{B.4.2}$$

induced by the bilinear map $(a/s) \times m \mapsto am/s$ is an isomorphism of A (and $S^{-1}A$) semi-modules. Furthermore it canonically induces a natural isomorphism between the endofunctors $S^{-1}A \otimes_A (-)$ and $S^{-1}(-)$

Proof. To prove that $S^{-1}A \otimes_A M \to S^{-1}M$ is an isomorphism it is enough to show that it gives a bijection of underlying sets. Surjectivity is clear, so it remains to prove injectivity. Note first that if $\sum(a_i/s_i) \otimes m_i$ is an element of $S^{-1}A \otimes_A M$ then if we set $s = \prod s_i$ and $t_i = \prod_{j \neq i} s_j$, we have

$$\sum_i (a_i/s_i)\otimes m_i = \sum_i (t_i a_i/s)\otimes m_i = \sum_i (1/s)\otimes(t_i a_i m_i) = (1/s)\otimes(\sum_i t_i a_i m_i) \tag{B.4.3}$$

hence every element of $S^{-1}A \otimes_A M$ is of the form $(a/s) \otimes m$. Now if $(a/s) \otimes m$ and $(b/t) \otimes n$ have the same image in $S^{-1}M$ then we have

$$am/s = bn/t \in S^{-1}M \tag{B.4.4}$$

hence there is some $u \in S$ such that

$$utam = usbn \in M. \tag{B.4.5}$$

Now we have

$$(a/s) \otimes m = (uta/stu) \otimes m = (1/stu) \otimes (usbn) = (usb/stu) \otimes n = (b/t) \otimes n, \tag{B.4.6}$$

showing injectivity. The final statement is now checked straightforwardly. □

Typically in this thesis we only extend scalars from $\mathbb{N}$ to a localization of this semi-ring. Lemma B.4.1 and Lemma B.2.6 tells us that in this case the extension of scalars functor will preserve monomorphisms.

Appendix C

Henselian rings and unibranched schemes

C.1 Henselian rings

In this section we state some standard results concerning Henselian rings and how any local (Noetherian) ring can be embedded in such a ring. Our main references are [Mil80] and [Stacks].

Henselian rings

Definition C.1.1 ([Stacks, Tag 04GF]). Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

1. We say R is *Henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.

2. We say R is *strictly Henselian* if R is Henselian and its residue field is separably algebraically closed.

The following lemma shows that there are many ways to check if a ring is Henselian;

Lemma C.1.2 ([Stacks, Tag 04GG]). *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent*

(1) *R is Henselian,*

(2) *for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,*

(3) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*

(4) *for any monic* $f \in R[T]$ *and any factorization* $\overline{f} = g_0 h_0$ *with* $\gcd(g_0, h_0) = 1$ *there exists a factorization* $f = gh$ *in* $R[T]$ *such that* $g_0 = \overline{g}$ *and* $h_0 = \overline{h}$ *and moreover* $\deg_T(g) = \deg_T(g_0)$,

(5) *for any* $f \in R[T]$ *and any factorization* $\overline{f} = g_0 h_0$ *with* $\gcd(g_0, h_0) = 1$ *there exists a factorization* $f = gh$ *in* $R[T]$ *such that* $g_0 = \overline{g}$ *and* $h_0 = \overline{h}$,

(6) *for any* $f \in R[T]$ *and any factorization* $\overline{f} = g_0 h_0$ *with* $\gcd(g_0, h_0) = 1$ *there exists a factorization* $f = gh$ *in* $R[T]$ *such that* $g_0 = \overline{g}$ *and* $h_0 = \overline{h}$ *and moreover* $\deg_T(g) = \deg_T(g_0)$,

(7) *for any étale ring map* $R \to S$ *and prime* $\mathfrak{q}$ *of* S *lying over* $\mathfrak{m}$ *with* $\kappa = \kappa(\mathfrak{q})$ *there exists a section* $\tau : S \to R$ *of* $R \to S$,

(8) *for any étale ring map* $R \to S$ *and prime* $\mathfrak{q}$ *of* S *lying over* $\mathfrak{m}$ *with* $\kappa = \kappa(\mathfrak{q})$ *there exists a section* $\tau : S \to R$ *of* $R \to S$ *with* $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$,

(9) *any finite* R*-algebra is a product of local rings,*

(10) *any finite* R*-algebra is a finite product of local rings,*

(11) *any finite type* R*-algebra* S *can be written as* $A \times B$ *with* $R \to A$ *finite and* $R \to B$ *not quasi-finite at any prime lying over* $\mathfrak{m}$,

(12) *any finite type* R*-algebra* S *can be written as* $A \times B$ *with* $R \to A$ *finite such that each irreducible component of* $\operatorname{Spec}(B \otimes_R \kappa)$ *has dimension* ≥ 1, *and*

(13) *any quasi-finite* R*-algebra* S *can be written as* $S = A \times B$ *with* $R \to A$ *finite such that* $B \otimes_R \kappa = 0$.

Corollary C.1.3. *Let* $(R, \mathfrak{m}, k)$ *be a Henselian ring. If* $R \to S$ *is a finite local homomorphism of local rings, then* S *is a Henselian local ring. In particular if* R *is Henselian and* J *is an ideal of* R*, then* R/J *is Henselian.*

A nice class of rings satisfying Hensels lemma is the complete local rings;

Lemma C.1.4 ([Stacks, Tag 04GM])**.** *Let* $(R, \mathfrak{m}, \kappa)$ *be a complete local ring, then* R *is Henselian.*

Example C.1.5. Recall that the ring of p-adic integers is the ring $\mathbb{Z}_p := \widehat{\mathbb{Z}_{(p)}}$. Its elements are formal power series of the form

$$\sum_{i=0}^{\infty} a_i p^i \ \ (0 \leq a_i \leq p - 1).$$

Since this ring is complete it is Henselian.

We now show how we can use the fact that the ring of 7-adic integers is Henselian to prove that 2 has a root in $\mathbb{Z}_7$.

Consider the polynomial

$$f(T) = T^2 - 2 \in \mathbb{Z}_7[T],$$

when reduced mod (7) we get the polynomial

$$\bar{f}(T) = T^2 - 2 \in \mathbb{F}_7$$

which has the roots ± 3, moreover since $\bar{f}'(3) = 6 \neq 0 \in \mathbb{F}_7$ we get from part (2) of Lemma C.1.2 that the root 3 lifts to an element $a \in \mathbb{Z}_7$ such that $f(a) = 0$ that is $a^2 = 2$.

Henselization

Definition C.1.6. Let R be a local ring. A *Henselization* of R is a local homomorphism of local rings $i : R \to R^h$ where R^h is a Henselian ring satisfying the following universal property: If $j : R \to H$ is a local homomorphism to a local Henselian ring, then there exists a unique morphism $j' : R^h \to H$ such that $j = j' \circ i$.

A *strict Henselization* of a local ring R is a local homomorphism of local rings $i : R \to R^{sh}$ such that any local homomorphism $j : R \to H$ with H strictly Henselian factors through $i : R \to R^{sh}$, moreover the factorisation is to be uniquely determined once the induced map $R^{sh}/\mathfrak{m}^{sh} \to H/\mathfrak{m}_H$ has been given.

For a local ring $(R, \mathfrak{m}, k)$ the Henselization R^h always exists. We provide the construction:

We follow [Mil80] and introduce the notion of an *étale neighborhood* of the local ring R. It is a pair $(B, \mathfrak{q})$ with B an étale R-algebra and $\mathfrak{q} \in \operatorname{Spec} B$ a prime ideal lying over $\mathfrak{m}$ such that $k(\mathfrak{q}) \cong k$. A morphism of étale neighborhoods $(B, \mathfrak{q}) \to (B', \mathfrak{q}')$ is an R-algebra map $\varphi : B \to B'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Using the tensor product one sees that the category of étale neighborhoods of R is a filtered category.

We set

$$R^h := \operatorname{colim}_{(B,q)} B.$$

The ring R^h consists of triples

$$(B, \mathfrak{q}, f)$$

with $(B, \mathfrak{q})$ an étale neighborhood of R and $f \in B$, and two such such triples $(B, \mathfrak{q}, f), (B', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists an étale neighborhood $(B'', \mathfrak{q}'')$ and morphisms of étale neighborhoods $\varphi : (B, \mathfrak{q}) \to (B'', \mathfrak{q}'')$ and $\varphi' : (B', \mathfrak{q}') \to (B, \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

The ring R^h satisfies the following properties:

Lemma C.1.7 ([Stacks, Tag 04GN])**.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties*

(1) *R^h is Henselian,*

(2) *R^h is a filtered colimit of étale R-algebras,*

(3) *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and*

(4) *$\kappa = R^h/\mathfrak{m}R^h$.*

Lemma C.1.8. *Suppose that I is an ideal of the local ring A. Then the Henselization of A/I is A^h/IA^h.*

Proof. By Corollary C.1.3 we have that the local ring A^h/IA^h is Henselian. Using the universal property of A^h we easily check that A^h/IA^h satisfies the universal property of $(A/J)^h$. □

Example C.1.9 ([Mil80, Ch. 1, Sec.4, Example 4.10])**.** Let k be a field, and let A be the localization of $k[T_1, \ldots, T_n]$ at $(T_1, \ldots, T_n)$. The Henselization of A is the subring of $k[[T_1, \ldots, T_n]]$ consisting of the formal power series which are algebraic over A.

The strict Henselization of a local ring $(R, \mathfrak{m}, k)$ also exists:

Lemma C.1.10 ([Stacks, Tag 04GP])**.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a commutative diagram*

$$\begin{tikzcd} \kappa \arrow[r] & \kappa \arrow[r] & \kappa^{sep} \\ R \arrow[u] \arrow[r] & R^h \arrow[u] \arrow[r] & R^{sh} \arrow[u] \end{tikzcd}$$

with the following properties

(1) *the map $R^h \to R^{sh}$ is local*

(2) *R^{sh} is strictly Henselian,*

(3) *R^{sh} is a filtered colimit of étale R-algebras,*

(4) *$\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and*

(5) *$\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.*

The strict Henselization R^{sh} given in lemma C.1.10 is constructed in a similar manner to that of the Henselization. The only difference being that in stead of étale neighborhoods, we use triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ is étale, $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : k(\mathfrak{q}) \to k^{sep}$ is an embedding of extensions of k.

More properties of the (strict) Henselization and reflected properties

We first state a bunch of results concerning the maps $R \to R^h \to R^{sh}$.

Lemma C.1.11 ([Stacks, Tag 07QM])**.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then we have the following*

1. $R \to R^h \to R^{sh}$ *are faithfully flat ring maps,*
2. $\mathfrak{m}R^h = \mathfrak{m}^h$ *and* $\mathfrak{m}R^{sh} = \mathfrak{m}^h R^{sh} = \mathfrak{m}^{sh}$*,*
3. $R/\mathfrak{m}^n = R^h/\mathfrak{m}^n R^h$ *for all n,*
4. *there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak{m}^n R^{sh}$ is a free $R/\mathfrak{m}^n$-module on $x_i \bmod \mathfrak{m}^n R^{sh}$.*

Lemma C.1.12 ([Stacks, Tag 07QN])**.** *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then*

1. $R \to R^h$*,* $R^h \to R^{sh}$*, and* $R \to R^{sh}$ *are formally étale,*
2. $R \to R^h$*,* $R^h \to R^{sh}$*, resp.* $R \to R^{sh}$ *are formally smooth in the* $\mathfrak{m}^h$*,* $\mathfrak{m}^{sh}$*, resp.* $\mathfrak{m}^{sh}$*-topology.*

Next we see how many properties of $(R, \mathfrak{m})$ are passed (and in some cases shared by) to R^h and R^{sh}.

Lemma C.1.13 ([Stacks, Tag 06LJ])**.** *Let R be a local ring. The following are equivalent*

1. *R is Noetherian,*
2. *R^h is Noetherian, and*
3. *R^{sh} is Noetherian.*

In this case we have

(a) $(R^h)^\wedge$ *and* $(R^{sh})^\wedge$ *are Noetherian complete local rings,*

(b) $R^\wedge \to (R^h)^\wedge$ *is an isomorphism,*

(c) $R^h \to (R^h)^\wedge$ *and* $R^{sh} \to (R^{sh})^\wedge$ *are flat,*

(d) $R^\wedge \to (R^{sh})^\wedge$ *is formally smooth in the* $\mathfrak{m}_{(R^{sh})^\wedge}$*-adic topology,*

(e) $(R^\wedge)^{sh} = R^\wedge \otimes_{R^h} R^{sh}$*, and*

(f) $((R^\wedge)^{sh})^\wedge = (R^{sh})^\wedge$*.*

Reducedness passes to the (strict) henselization.

Lemma C.1.14 ([Stacks, Tag 06DH]). *Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.*

Lemma C.1.15 ([Stacks, Tag 06DI]). *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.*

Lemma C.1.16 ([Stacks, Tag 06LK]). *Given any local ring R we have $\dim(R) = \dim(R^h) = \dim(R^{sh})$.*

Lemma C.1.17 ([Stacks, Tag 06LN]). *Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.*

Lemma C.1.18 ([Stacks, Tag 06LN]). *Let R be a Noetherian local ring. Then R is a discrete valuation ring if and only if R^h is a discrete valuation ring if and only if R^{sh} is a discrete valuation ring.*

C.2 Unibranched schemes

Intuitively a unibranched scheme is a scheme which when considered "very locally" is irreducible. For instance the nodal cubic X given by the vanishing of $(y^2 - x^2 - x^3)$ in $\mathbb{A}^2_k$ (with $k = \bar{k}, \operatorname{char} k \neq 2$) is irreducible, however when we consider the geometric picture we see that near the origin X looks like two lines crossing, and hence "very locally" this should not be irreducible. The aforementioned example of the nodal cubic will be a scheme which is not unibranched.

Unibranched rings

Definition C.2.1 ([GD67, Chapter 0 (23.2.1)]). Let A be a local ring. We say A is *unibranched* if the reduction A_{red} is a domain and if the integral closure A' of A_{red} in its field of fractions is local. We say A is *geometrically unibranched* if A is unibranched and moreover the residue field of A' is purely inseparable[1] over the residue field of A.

Remark C.2.2. Let A be a local ring. Here is an equivalent formulation

(1) A is unibranched if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring, and

(2) A is geometrically unibranched if A has a unique minimal prime $\mathfrak{p}$ and the integral closure of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.

[1]Note that by definition the trivial field extension is a purely inseparable extension.

Example C.2.3. It follows from the definitions that a normal local ring is geometrically unibranched.

Example C.2.4. Let X be the nodal cubic $\operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$. Let $x = (x,y)$ denote the origin. Then we claim that the local ring $\mathcal{O}_{X,x}$ is not unibranched. To see this consider first the ring map

$$k[x,y] \to k[t]$$

given by

$$x \mapsto t^2 - 1;\ y \mapsto (t^2-1)t = t^3 - t.$$

We easily see that $(y^2 - x^2 - x^3)$ is the kernel of the aforementioned morphism. Hence we have that $\mathcal{O}_X(X) \cong k[t^2-1, t^3-t] = A$ and the ideal $\mathfrak{m} = (t^2-1, t^3-t)$ corresponds to the ideal $(x,y) \in k[x,y]/(y^2-x^2-x^3)$ under this isomorphism. Thus we need to show that the integral closure of the local ring $A_\mathfrak{m}$ in its field of fractions $k(t)$ is not local. For this purpose note that the integral closure of A in $k(t)$ is nothing but $k[t]$ and since integral closure commutes with localization we have that the integral closure of $A_\mathfrak{m}$ is the ring $S^{-1}k[t]$ where $S = A \setminus \mathfrak{m}$. Since

$$(t-1)k[t] \cap A = (t^2-1, t^3-t) = \mathfrak{m} = (t+1)k[t] \cap A$$

it follows that $S^{-1}k[t]$ is not local.

The notion of unibranchedness can also be understood in terms of Henselisation:

Lemma C.2.5 ([GD67, Chapter IV Proposition 18.6.12])**.** *Let A be a local ring. Let A^h be the henselization of A. The following are equivalent*

(1) *A is unibranched, and*

(2) *A^h has a unique minimal prime.*

Lemma C.2.6 ([GD67, Chapter IV Proposition 18.8.15])**.** *Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent*

(1) *A is geometrically unibranched, and*

(2) *A^{sh} has a unique minimal prime.*

Definition C.2.7 ([Stacks, Tag 0C26])**.** Let A be a local ring with henselization A^h and strict henselization A^{sh}. The *number of branches of A* is the number of minimal primes of A^h if finite and ∞ otherwise. The *number of geometric branches of A* is the number of minimal primes of A^{sh} if finite and ∞ otherwise.

Lemma C.2.8 ([Stacks, Tag 0C37])**.** *Let $(A, \mathfrak{m}, \kappa)$ be a local ring.*

(1) *If A has infinitely many minimal prime ideals, then the number of (geometric) branches of A is ∞.*

(2) *The number of branches of A is* 1 *if and only if A is unibranched.*

(3) *The number of geometric branches of A is* 1 *if and only if A is geometrically unibranched.*

Assume A has finitely many minimal primes and let A' be the integral closure of A in the total ring of fractions of A_{red}. Then

(4) *the number of branches of A is the number of maximal ideals $\mathfrak{m}'$ of A',*

(5) *to get the number of geometric branches of A we have to count each maximal ideal $\mathfrak{m}'$ of A' with multiplicity given by the separable degree of $\kappa(\mathfrak{m}')/\kappa$.*

Branches of the completion.

It can be hard to compute integral closures and (strict) Henselizations. We now describe how (geometrically) unibranchedness can to some extent be checked by studying completions instead.

Lemma C.2.9 ([Stacks, Tag 0C28])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring.*

(1) *The map $A^h \to A^\wedge$ defines a surjective map from minimal primes of $A^\wedge$ to minimal primes of A^h.*

(2) *The number of branches of A is at most the number of branches of $A^\wedge$.*

(3) *The number of geometric branches of A is at most the number of geometric branches of $A^\wedge$.*

For one dimensional local Noetherian rings the number of branches corresponds to the number of minimal primes of the completion.

Lemma C.2.10 ([Stacks, Tag 0C2D])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension* 1*. Then the number of (geometric) branches of A and $A^\wedge$ is the same.*

Actually we have a smiliar result for excellent Noetherian local rings.

Lemma C.2.11 ([Stacks, Tag 0C2E], [Bed13, Thm 2.3])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring. If the formal fibres of A are geometrically normal (for example if A is excellent or quasi-excellent), then A is Nagata and the number of (geometric) branches of A and $A^\wedge$ is the same.*

Unibranched schemes

We now provide the obvious scheme theoretic versions of the definitions and results of the previous section.

Definition C.2.12 ([GD67, Chapter IV (6.15.1)]). Let X be a scheme. Let $x \in X$. We say X is *unibranched at* x if the local ring $\mathcal{O}_{X,x}$ is unibranched. We say X is *geometrically unibranched at* x if the local ring $\mathcal{O}_{X,x}$ is geometrically unibranched. We say X is *unibranched* if X is unibranched at all of its points. We say X is *geometrically unibranched* if X is geometrically unibranched at all of its points.

Remark C.2.13. Note that being unibranched for a local ring A (Definition C.2.1) is a condition on the ring A, while for a scheme it is a condition on all of the stalks of the scheme. Hence it is possible that the local ring A is unibranched, but the scheme $\mathrm{Spec}(A)$ is not unibranched as a scheme.

Lemma C.2.14. *A scheme is unibranched (resp. geometrically unibranched) if and only if* $\mathrm{Spec}\, \mathcal{O}^h_{X,x}$ *(resp.* $\mathrm{Spec}\, \mathcal{O}^{sh}_{X,x}$*)) is irreducible for every* $x \in X$*.*

Proof. This follows from Lemma C.2.5 (resp. C.2.6). □

Remark C.2.15. Lemma C.2.14 tells us that the definition of a (gemetrically) unibranched scheme given here coincides with the one given in [SV00, Def.2.1.5].

Proposition C.2.16 ([SV00, Prop.2.1.6]). *Let S be a Noetherian geometrically unibranch scheme and $f : S' \to S$ be a proper birational morphism. Then for any point s of S the fiber S'_s of f over s is geometrically connected.*

Proof. It follows from [GD63, (4.3.5)] and [GD67, (18.8.15(c))]. □

The following theorem shows that all Normal schemes are necessarily geometrically unibranched.

Theorem C.2.17. *The following are equivalent:*

(1) *A scheme X is unibranched (resp. geometrically unibranched)*

(2) *The scheme* $\mathrm{Spec}(\mathcal{O}^h_{X,x})$ *(resp.* $\mathrm{Spec}(\mathcal{O}^{sh}_{X,x})$*) is irreducible for all points $x \in X$ where the local ring $\mathcal{O}_{X,x}$ is not normal.*

Proof. Using Lemma C.2.5 (resp. C.2.6) we get that (1) implies (2). The converse follows from Lemma C.1.15. □

Definition C.2.18 ([Stacks, Tag 0C38]). Let X be a scheme. Let $x \in X$. The *number of branches of X at x* is the number of branches of the local ring $\mathcal{O}_{X,x}$ as defined in Definition C.2.7. The *number of geometric branches of X at x* is the number of geometric branches of the local ring $\mathcal{O}_{X,x}$ as defined in Definition C.2.7.

From Lemma C.2.8 we obtain the following result:

Lemma C.2.19. *Let X be a scheme. Let $x \in X$.*

(1) *The number of branches of X at x is 1 if and only if X is unibranched at x.*

(2) *The number of geometric branches of X at x is 1 if and only if X is geometrically unibranched at x.*

Lemma C.2.20. *Suppose X is an integral locally Noetherian scheme of dimension one. Then X is (geometrically) unibranched if and only if $\widehat{\mathcal{O}_{X,x}}$ is for every $x \in X$. In particular we have that X is unibranched if and only if* $\operatorname{Spec}(\widehat{\mathcal{O}_{X,x}})$ *is irreducible.*

Proof. The first assertion follows from Lemma C.2.10. The final assertion follows from the first part together with C.2.5 and the fact that the henselization of a Henselian ring is itself. $\square$

Similarly we have that

Lemma C.2.21. *Suppose X is an excellent scheme. Then X is (geometrically) unibranched if and only if $\widehat{\mathcal{O}_{X,x}}$ is for every $x \in X$ with $\mathcal{O}_{X,x}$ not being a normal ring. In particular we have that X is unibranched if and only if* $\operatorname{Spec}(\widehat{\mathcal{O}_{X,x}})$ *is irreducible.*

Proof. This follows Theorem C.2.17 together with Lemma C.2.11 and reasoning similar to that of the proof given of Lemma C.2.20. $\square$

Examples

We saw in Lemma C.2.21 that for excellent schemes such as for instance algebraic schemes, we can check unibranchedness by checking if the scheme $\operatorname{Spec}(\widehat{\mathcal{O}_{X,x}})$ is irreducible at the non-normal points. The following lemma will help us check the latter condition.

Lemma C.2.22. *Let $(A, \mathfrak{m})$ be a local Noetherian ring, and $\mathfrak{a}$ be an ideal of A contained in the maximal ideal $\mathfrak{m}$. Then we have an isomorphism of $\mathfrak{m}$-adic completions*

$$\hat{A}/\mathfrak{a}\hat{A} \cong \widehat{(A/\mathfrak{a})}.$$

Proof. Since A is Noetherian we have an isomorphism

$$\mathfrak{a} \otimes_A \hat{A} \to \hat{\mathfrak{a}}$$

hence the image of $\hat{\mathfrak{a}}$ in $\hat{A}$ is the ideal $\mathfrak{a}\hat{A}$. The result now follows from the exact sequence

$$0 \to \hat{a} \to \hat{A} \to \widehat{(A/\mathfrak{a})} \to 0.$$

$\square$

We will now prove again using different methods that the plane nodal cubic curve is not unibranched in the case where the characteristic of the base field is different from 2.

Example C.2.23. The plane nodal cubic curve $X = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2_k$ is not unibranched.

Proof. We only need to check the condition on the singular point which is the origin. By Lemma C.2.21 we can prove that X is not unibranched by showing that the ring $(k[x,y]/\widehat{(y^2 - x^3} - x^2))_{(x,y)}$ has more than one minimal prime ideal. Note that by Lemma C.2.22 we have an isomorphism

$$(k[x,y]/\widehat{(y^2 - x^3} - x^2))_{(x,y)} \cong k[[x,y]]/(y^2 - x^3 - x^2).$$

We shall now construct two formal power series

$$g = y + x + \sum_{i=2}^{\infty} g_i$$
$$h = y - x + \sum_{j=2}^{\infty} h_j$$

where g_i, h_i are homogeneous of degree i and such that $gh = y^2 - x^3 - x^2$. This is done step by step. First note that $(y+x)(y-x) = y^2 - x^2$, next note that since $(y+x)$ and $(y-x)$ generate the maximal ideal of $k[[x,y]]$[2] we can find polynomials g_2, h_2 such that

$$(y+x)h_2 + (y-x)g_2 = -x^3.$$

Indeed we can take $h_2 = \frac{1}{2}x^2$ and $g_2 = \frac{-1}{2}x^2$. We have almost got what we want except we need to get rid of a term g_2h_2, so again we can find polynomials h_3, g_3 of degree 3 such that

$$(y+x)h_3 + (y-x)g_3 = -g_2h_2.$$

We now want to get rid of $h_3g_2 + g_3h_2$ and this can be done in a similar fashion, and continuing in this way we get the formal power series g and h.

Now since g, h begin with linearly independent terms and all the other terms are homogeneous of degree i for $i = 2, \ldots, \infty$, it follows that g, h generate the maximal ideal of $k[[x,y]]$, thus we have an automorphism $k[[x,y]] \to k[[x,y]]$ given by $x \mapsto g$, $y \mapsto h$, hence we have

$$k[[x,y]]/(y^2 - x^2 - x^3) \cong k[[x,y]]/(gh) \cong k[[x,y]]/(xy)$$

which clearly has at least two minimal prime ideals. $\square$

[2] Here we are using the assumption that $\operatorname{char} k \neq 2$.

Appendix D

Sheaves in Grothendieck topologies

D.1 Grothendieck pretopologies

We recall the basic definitions and constructions following [Fan+05] and [Vis04].

Definition D.1.1. Let $\mathcal{C}$ be a category. A Grothendieck pretopology τ on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ a collection of sets of arrows $\{U_i \to U\}$, called *coverings* of U, so that the following conditions are satisfied.

(i) If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering.

(ii) If $\{U_i \to U\}$ is a covering and $V \to U$ is any arrow, then the fibered products $\{U_i \times_U V\}$ exist, and the collection of projections $\{U_i \times_U V \to V\}$ is a covering.

(iii) If $\{U_i \to U\}$ is a covering, and for each index i we have a covering $\{V_{i,j} \to U_i\}$ (here j varies on a set depending on i), the collection of composites $\{V_{i,j} \to U_i \to U\}$ is a covering of U.

We denote the collection of coverings of an object $U \in \mathcal{C}$ by $\mathrm{Cov}(U)$. A category with a Grothendieck pretopology is called a *site* and we denote it by the pair $(\mathcal{C}, \tau)$. For a Grothendieck pretopology τ on $\mathcal{C}$ we will also sometimes use the notation $\mathrm{Cov}_\tau(U)$ in place of $\mathrm{Cov}(U)$. This notation is convenient when we need to talk about more than one Grothendieck pretopology on a given category.

Definition D.1.2. Let $(\mathcal{C}, \tau)$ be a site and suppose that $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$. A *refinement* of $\{U_i \to U\}_{i \in I}$ is a set of maps $\{V_a \to U\}_{a \in A}$ such that for each index $a \in A$ there is some index $i \in I$ such that $V_a \to U$ factors through $U_i \to U$.

If τ_1, τ_2 are two Grothendieck pretopologies on a category $\mathcal{C}$. Then we say that τ_1 is *finer* than τ_2 (and τ_2 is *coarser* than τ_1) if any covering $\mathcal{U} \in \mathrm{Cov}_{\tau_2}(U)$ there is a refinement $\mathcal{V} = \{V_a \to U\}_{a \in A}$ of $\mathcal{U}$ with $\mathcal{V} \in \mathrm{Cov}_{\tau_1}(U)$.

D.2 Sheaves on a site

For a category $\mathcal{C}$ we denote the category of presheaves on $\mathcal{C}$ by $\mathrm{Psh}(\mathcal{C})$.

Let $(\mathcal{C}, \tau)$ be a site, $\mathcal{F} \in \mathrm{Psh}(\mathcal{C})$. Given a covering $\{p_i : U_i \to U\}_{i \in I}$. Let $a : \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ be the unique map such that

$$\mathcal{F}(p_i) = pr_i \circ a$$

where $pr_j : \prod_{i \in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j)$ is projection onto the j'th factor. Further for each ordered pair of indices (i, j) let $pr_{1,i,j} : U_i \times_U U_j \to U_i$ be the projection to U_i and $pr_{2,i,j} : U_i \times_U U_j \to U_j$ be the projection to U_j. Let b_1, b_2 be the unique maps $b_1, b_2 \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ such that

$$pr_{(i,j)} \circ b_1 = \mathcal{F}(pr_{1,i,j}) \circ pr_i\,; \tag{D.2.1}$$

$$pr_{(i,j)} \circ b_2 = \mathcal{F}(pr_{2,i,j}) \circ pr_j \tag{D.2.2}$$

for each i, j where $pr_{(i,j)}$ is the projection onto the (i, j)'th factor

$$pr_{(i,j)} : \prod_{l,m} \mathcal{F}(U_l \underset{U}{\times} U_m) \to \mathcal{F}(U_i \underset{U}{\times} U_j).$$

Definition D.2.1. A presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Sets}$ is a *separated* presheaf if for every covering $\{U_i \to U\}_{i \in I}$ the map

$$\mathcal{F}(U) \overset{a}{\to} \prod_{i \in I} \mathcal{F}(U_i)$$

is injective. It is a *sheaf* if the following diagram

$$\mathcal{F}(U) \xrightarrow{\ a\ } \prod_{i \in I} \mathcal{F}(U_i) \underset{b_2}{\overset{b_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer. We denote the category of sheaves by $\mathrm{Sh}(\mathcal{C})$.

Since a category can have many different Grothendieck pretopologies it is convenient to state which site we are considering when speaking of a sheaf. Hence if τ is a Grothendieck pretopology on $\mathcal{C}$ we say that the presheaf $\mathcal{F}$ on $\mathcal{C}$ is a *τ-sheaf* if it is a sheaf on the site $(\mathcal{C}, \tau)$.

Definition D.2.2. ([Vis04, Def. 2.52]) A pretopology $\mathcal{T}$ on a category $\mathcal{C}$ is called *saturated* if a set of arrows $\{U_i \to U\}$ which has a refinement that is in $\mathcal{T}$ is also in $\mathcal{T}$. If $\mathcal{T}$ is a pretopology on $\mathcal{C}$, the *saturation* $\overline{\mathcal{T}}$ of $\mathcal{T}$ is the set of sets of arrows which have a refinement in $\mathcal{T}$.

Proposition 2.53 in [Vis04] tells us that the saturation of a pretopology is saturated and that a presheaf is a sheaf with respect to $\mathcal{T}$ if and only if it is a sheaf with respect to the saturation of $\mathcal{T}$.

Definition D.2.3 ([Vis04, Def.2.57])**.** A pretopology τ on $\mathcal{C}$ is called *subcanonical* if every representable functor on $\mathcal{C}$ is a sheaf with respect to τ.

Remark D.2.4. As mentioned in the second paragraph following [Vis04, Def.2.57] the name "subcanonical" comes from the fact that on a category $\mathcal{C}$ there is a topology, known as the *canonical topology*, which is the finest topology in which every representable functor is a sheaf.

Sheafification

Definition D.2.5. Let $(\mathcal{C}, \tau)$ be a site, and $\mathcal{F} \in \mathrm{Psh}(\mathcal{C})$. A *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}_\tau \in \mathrm{Sh}(\mathcal{C})$, together with a natural transformation $\mathcal{F} \to \mathcal{F}_\tau$, such that

(i) given an object U of $\mathcal{C}$ and two elements ξ and η of $\mathcal{F}(U)$ whose images ξ^a and η^a in $\mathcal{F}_\tau(U)$ are the same, there exists a covering $\{\sigma_i : U_i \to U\}$ such that $\sigma_i^*(\xi) := \mathcal{F}(\sigma_i)(\xi) = \sigma_i^*(\eta)$, and

(ii) for each object U of $\mathcal{C}$ and each $\overline{\xi} \in \mathcal{F}_\tau(U)$, there exists a covering $\{\sigma_i : U_i \to U\}$ and elements $\xi_i \in \mathcal{F}(U_i)$ such that $\xi_i^a = \sigma_i^*\overline{\xi}$.

Theorem D.2.6. *Let $(\mathcal{C}, \tau)$ be a site, $\mathcal{F} \in \mathrm{Psh}(\mathcal{C})$.*

(i) *If $\mathcal{F}_\tau \in \mathrm{Sh}(\mathcal{C})$ is a sheafification of $\mathcal{F}$, any morphism from $\mathcal{F}$ to a sheaf factors uniquely through $\mathcal{F}_\tau$.*

(ii) *There exists a morphism $\mathcal{F} \to \mathcal{F}_\tau^s$ where $\mathcal{F}_\tau^s$ is a separated presheaf, such that any morphism from $\mathcal{F}$ to a separated presheaf factors uniquely through $\mathcal{F}_\tau^s$.*

(iii) *There exists a sheafification $\mathcal{F} \to \mathcal{F}_\tau$, which is unique up to a canonical isomorphism.*

(iv) *The natural transformation $\mathcal{F} \to \mathcal{F}_\tau$ is injective if and only if $\mathcal{F}$ is separated.*

Sketch of proof. Part (iv) follows easily from the definition. For (i) and (ii) and (iii) we only provide the constructions. **For (i):** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a natural transformation from $\mathcal{F}$ to a sheaf $\mathcal{G} \in \mathrm{Sh}(\mathcal{C})$. Given an element $\overline{\xi} \in \mathcal{F}_\tau(U)$ we want to define the image of ξ in $\mathcal{G}(U)$. There exists a covering $\{\sigma_i : U_i \to U\}$ and elements $\xi_i \in \mathcal{F}(U_i)$, such that the image of ξ_i in $\mathcal{F}_\tau(U_i)$ is $\sigma_i^*(\overline{\xi})$. Set $\eta_i = \phi(U_i)(\xi_i) \in \mathcal{G}(U_i)$. The pullbacks $pr_1^*\xi_i$ and $pr_2^*\xi_j$ in $\mathcal{F}(U_i \times_U U_j)$ both have as their image in $\mathcal{F}_\tau(U_i \times_U U_j)$ the pullback of $\overline{\xi}$; hence there is a covering

$\{U_{i,j,\alpha} \to U_i \times_U U_j\}_\alpha$ such that ξ_i and ξ_j both pullback to the same element in $\mathcal{F}(U_{i,j,\alpha})$ for every α. Using this together with the fact that $\mathcal{G}$ is separated we get that η_i and η_j both pullback to the same element in $\mathcal{G}(U_i \times_U U_j)$ and since $\mathcal{G}$ is a sheaf we get that the η_i glue to give an element $\eta \in \mathcal{G}(U)$. We now let $\phi_\tau : \mathcal{F}_\tau \to \mathcal{G}$ be given by $\phi_\tau(U)(\overline{\xi}) = \eta$.

For (ii): For each object U of $\mathcal{C}$, we define an equivalence relation $\sim$ on $\mathcal{F}(U)$ as follows: Given two elements a and b of $\mathcal{F}(U)$, we write $a \sim b$ if there is a covering $\{U_i \to U\}_{i \in I}$ such that the pullbacks of a and b to each U_i coincide. We define $\mathcal{F}^s_\tau(U) := \mathcal{F}(U)/\sim$. If $V \to U$ is a morphism in $\mathcal{C}$, then the pullback $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the equivalence relations, yielding a pullback $\mathcal{F}^s_\tau(U) \to \mathcal{F}^s_\tau(V)$. This defines the functor $\mathcal{F}^s_\tau$ with the surjective morphism $\mathcal{F} \to \mathcal{F}^s_\tau$. For an element $a \in \mathcal{F}(U)$ we denote its image in $\mathcal{F}^s_\tau(U)$ by $[a]$ and since $\mathcal{F} \to \mathcal{F}^s_\tau$ is surjective we denote any element in $\mathcal{F}^s_\tau(U)$ in this way. The presheaf $\mathcal{F}^s_\tau$ is separated, and every natural transformation from $\mathcal{F}$ to a separated presheaf factors uniquely through $\mathcal{F}^s_\tau$.

For (iii): To construct $\mathcal{F}_\tau$, we take for each object U of $\mathcal{C}$ the set of pairs $(\{U_i \to U\}, \{[a_i]\})$, where $\{U_i \to U\}$ is a covering, and $\{[a_i]\}$ is a set of elements with $[a_i] \in \mathcal{F}^s_\tau(U_i)$, such that the pullback of $[a_i]$ and $[a_j]$ to $\mathcal{F}^s_\tau(U_i \times_U U_j)$, along the first and second projection coincide. On this set we impose an equivalence relation, by declaring $(\{U_i \to U\}, \{[a_i]\})$ to be equivalent to $(\{V_j \to U\}, \{[b_j]\})$ when the restrictions of $[a_i]$ and $[b_j]$ to $F^s(U_i \times_U V_j)$, along the first and second projection respectively, coincide. For each U, we denote by $\mathcal{F}_\tau(U)$ the set of equivalence classes. If $V \to U$ is a morphism, we define a function $\mathcal{F}_\tau(U) \to \mathcal{F}_\tau(V)$ by associating with the class of a pair $(\{U_i \to U\}, \{[a_i]\})$ in $\mathcal{F}_\tau(U)$ the class of the pair $(\{U_i \times_U V\}, p_i^*([a_i])$, where $p_i : U_i \times_U V \to U_i$ is the projection. This gives a sheaf $\mathcal{F}_\tau$. There is also a natural transformation $\mathcal{F}^s_\tau \to \mathcal{F}_\tau$, obtained by sending an element $[a] \in \mathcal{F}^s_\tau(U)$ into $(\{U = U\}, [a])$ and the composition

$$\mathcal{F} \to \mathcal{F}^s_\tau \to \mathcal{F}_\tau$$

is the sheafification. □

Remark D.2.7. Note that $\mathcal{F}_\tau(U) = \operatorname{colim}_{\mathcal{V}} \check{H}^0(\mathcal{V}, \mathcal{F}^s_\tau)$ where $\mathcal{V}$ is a covering, $\check{H}^0(\mathcal{V}, \mathcal{F}^s_\tau)$ denotes the equalizer of the obvious diagram and the colimit is a filtered colimit ordered by the relation "refinement".

Remark D.2.8. (Explicit computation of the induced morphism from the sheafification). Suppose $\eta : \mathcal{F} \to \mathcal{G}$ is a morphism from the presheaf $\mathcal{F}$ to the sheaf $\mathcal{G}$. Let $\mathcal{F}_\tau$ be the sheafification as constructed in the proof of Theorem D.2.6 and let $\mathrm{Sh}(\eta)$ denote the induced map $\mathrm{Sh}(\eta) : \mathcal{F}_\tau \to \mathcal{G}$. Then for any element $f = (\{U_i \to X\}_{i \in I}, \overline{f_i} \in \mathcal{F}^s_\tau(U_i)) \in \mathcal{F}_\tau(X)$, where $\overline{f_i}$ is the image of $f_i \in \mathcal{F}(U_i)$ in $\mathcal{F}^s_\tau(U_i)$ we have that $\mathrm{Sh}(\eta)(X)(f)$ is the "gluing of the elements" $\eta(U_i)(f_i)$.

Remark D.2.9. Suppose that $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves. Then the induced morphism

$$\phi_\tau : \mathcal{F}_\tau \to \mathcal{G}_\tau$$

is given by

$$\phi_\tau(U)(\{p_i : U_i \to U\}, \{[a_i]\}) = (\{p_i : U_i \to U\}, \{[\phi(U_i)(a_i)]\})$$

Definition D.2.10. (Notation and assumptions as in Theorem D.2.6) We will call the separated presheaf $\mathcal{F}^s_\tau$ constructed in the proof of Theorem D.2.6 the *separation of the presheaf* $\mathcal{F}$.

Definition D.2.11. Let $(\mathcal{C}, \tau)$ be a site and let $\phi : F \to G$ be a morphism of presheaves on $\mathcal{C}$.

1. The morphism ϕ is a *τ-local monomorphism* (or *τ-locally a monomorphism*) if for any $U \in \mathcal{C}$ and $a, b \in F(U)$ such that $\phi(U)(a) = \phi(U)(b) \in G(U)$ there exists a τ-covering $\{p_i : U_i \to U\}_{i \in I}$ such that $p_i^*(a) = p_i^*(b)$ for all $i \in I$.

2. The morphism ϕ is a *τ-local epimorphism* (or *τ-locally an epimorphism*) if for any $U \in \mathcal{C}$ and $b \in G(U)$ there exists a τ-covering $\{p_i : U_i \to U\}_{i \in I}$ and elements $a_i \in F(U_i)$ such that

   $$p_i^*(b) = \phi(U_i)(a_i)$$

   for all i.

3. The morphism ϕ is a *τ-local isomorphism* (or *τ-locally an isomorphism*) if it is both a τ-local monomorphism and a τ-local epimorphism.

Lemma D.2.12. *Let $(\mathcal{C}, \tau)$ be a site and $\phi : F \to G$ a morphism of presheaves on $\mathcal{C}$. Then the induced morphism $(\phi)_\tau : F_\tau \to G_\tau$ is an isomorphism if and only if ϕ is a τ-local isomorphism.*

Definition D.2.13. Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We say that $\mathcal{F}$ is *representable* with respect to the topology τ (or *τ-representable*) if there exists an object $X \in \mathcal{C}$ such that the sheafification of $\mathcal{F}$ is isomorphic to the sheafification of the representable presheaf h_X.

D.3 Grothendieck topologies

The definitions of the previous section can be rephrased more concisely in terms of Grothendiecks topologies. Every Grothendieck pretopology has an associated Grothendieck topology and two different Grothendieck pretopologies can give the same Grothendieck topology, but if this happens the sheaves with respect to the two different pretopologies will coincide.

This section is mostly written for the curious reader and is not really needed to understand the contents of this thesis.

Most of the definitions stated here are taken directly from [MM94, Ch. III, Sec.2].

Definition D.3.1. Let $\mathcal{C}$ be a category. For an object $U \in C$ a *sieve* on U is a subfunctor $S \subset h_U$ of the presheaf represented by U.

For an arrow $f : V \to U$ in $\mathcal{C}$ and a sieve S on U we have an induced sieve on V denoted by f^*S given by the formula:

$$f^*S(W) := \{g \in \mathrm{h}_V(W) \,:\, f \circ g \in S(W)\} \tag{D.3.1}$$

which we call the *restriction of* S along f.

Definition D.3.2. A *Grothendieck topology* J on a category $\mathcal{C}$ is the data consisting of a class $J(U)$ of sieves on U for every object U in $\mathcal{C}$ subject to the following conditions:

1. For every object $U \in \mathcal{C}$ we have $h_U \in J(U)$.
2. If $S \in J(U)$ and $f : V \to U$ is any arrow in $\mathcal{C}$ then $f^*S \in J(V)$.
3. If $S \in J(U)$ and R is any sieve on U such that $f^*R \in J(V)$ for all $f \in S(V)$, then $R \in J(C)$.

These three axioms are called the *maximality*, *stability* and *transitivity* axioms respectively. If $S \in J(U)$ we say that S is a *covering sieve*.

Definition D.3.3. Let $\mathcal{C}$ be a category with a Grothendieck pretopology τ. For a covering $\mathcal{U} = \{p_i : U_i \to U\}_{i \in I}$ of an object $U \in \mathcal{C}$ we let $h_{\mathcal{U}} \subset h_U$ be the sieve on U defined by taking $h_{\mathcal{U}}(T)$ to be the set of morphisms $f : T \to U$ with the property that for some $i \in I$ there is a factorization $f = (T \to U_i \xrightarrow{p_i} U)$. The sieve $h_{\mathcal{U}}$ is called the *sieve associated with the covering* $\mathcal{U}$.

Proposition D.3.4. *Let $\mathcal{C}$ be a category with a Grothendieck pretopology τ. For every object $U \in \mathcal{C}$ let $J_\tau(U)$ be the class of sieves S on U with the property that there is some covering $\mathcal{U} \in \mathrm{Cov}(U)$ such that $h_{\mathcal{U}} \subset S$. Then the classes $J_\tau(U)$ form a Grothendieck topology J_τ on $\mathcal{C}$.*

Proof. See [MM94, Ch. III, Sec.2, page 112]. □

Definition D.3.5. For a category $\mathcal{C}$ with a Grothendieck pretopology τ the Grothendieck topology J_τ from Proposition D.3.4 is called the *Grothendieck topology associated to* τ.

Definition D.3.6. Let $\mathcal{C}$ be a category with a Grothendieck topology J. A presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Sets}$ is a *J-sheaf* (or *sheaf in the J-topology*) if for any object $U \in \mathcal{C}$ and covering sieve $S \in J(U)$ the inclusion $S \subset h_U$ induces a bijection

$$\mathrm{Hom}(h_U, \mathcal{F}) \to \mathrm{Hom}(S, \mathcal{F}). \tag{D.3.2}$$

Proposition D.3.7. *Let $\mathcal{C}$ be a category with a Grothendieck pretopology τ with associated Grothendieck topology J_τ and $\mathcal{F} : \mathcal{C}^{op} \to \mathsf{Sets}$ a presheaf on $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf with respect to τ (Definition D.2.1) if and only if $\mathcal{F}$ is a J_τ-sheaf (Definition D.3.6).*

Proof. See [Vis04, Prop.2.42]. □

D.4 Examples

Let us give some examples of Grothendieck pretopologies which are often encountered in algebraic geometry.

Example D.4.1.

(A topological space) If X is a topological space, then letting $\mathcal{C}_X$ be the category where objects are the open subsets of X and the morphisms are given by inclusions. Then setting $\mathrm{Cov}(U)$ to be the open coverings of the open subset U we get a Grothendieck pretopology τ_X. This follows easily from the fact that the fiber product of two open subsets is their intersection.

If X is a scheme considered as a topological space and $\mathcal{F}$ is a presheaf on $\mathcal{C}_X$ we say that $\mathcal{F}$ is a *Zariski sheaf* on X if $\mathcal{F}$ is a sheaf on the site $(\mathcal{C}_X, \tau_X)$.

(The Zariski topology) For a scheme U taking $\mathrm{Cov}(U)$ to consist of exactly those sets of the form $\{U_i \xrightarrow{p_i} U\}_{i\in I}$ where p_i are open embeddings and we have $X = \cup_{i\in I} p_i(U_i)$ gives a Grothendieck pretopology on the category of schemes whose associated Grothendieck topology is called the *Zariski topology* on the category of schemes.

A sheaf in the Zariski topology is called a *Zariski sheaf*.

(The Étale topology) Taking coverings $\{U_i \to U\}$ to be jointly surjective families of étale morphisms we get a pretopology whose associated Grothendieck topology is called the *étale topology* on the category of schemes.

A sheaf in the étale topology is called an *étale sheaf*.

(The fppf topology) The *fppf topology* on the category of schemes is the Grothendieck topology associated to the pretopology where coverings $\{U_i \to U\}$ is a jointly surjective family of flat maps locally of finite presentation.

Appendix E

Monoid objects

E.1 Monoid objects

Definition E.1.1. A *commutative monoid object* in a category $\mathcal{C}$ with products and a terminal object $*$ is an object M in $\mathcal{C}$ and arrows

$$0 : * \to M$$

(the zero map)

$$+ : M \times M \to M$$

(the addition map), such that the following diagrams commute:

$$\begin{tikzcd} M \times M \arrow[r, "+"] \arrow[d, "swap"] & M \arrow[d, equal] \\ M \times M \arrow[r, "+"] & M. \end{tikzcd} \tag{E.1.1}$$

(expressing commutativity of addition)

$$\begin{tikzcd} M \times M \times M \arrow[r, "M\times +"] \arrow[d, "+\times M"] & M \times M \arrow[d, "+"] \\ M \times M \arrow[r, "+"] & M. \end{tikzcd} \tag{E.1.2}$$

(yielding associativity of addition)
and

$$\begin{tikzcd} M \arrow[r, "\cong"] \arrow[d, "\cong"] \arrow[drr, equal] & * \times M \arrow[r, "0\times M"] & M \times M \arrow[d, "+"] \\ M \times * \arrow[r, "M\times 0"] & M \times M \arrow[r, "+"] & M \end{tikzcd} \tag{E.1.3}$$

(informally telling us that the zero map picks out an element that is a left and right identity).

Presheaves represented by monoid objects

Suppose that M together with the arrows $0 : * \to M$, $+ : M \times M \to M$ is a monoid object in a category $\mathcal{C}$ with products and terminal object $*$. This gives rise to a monoid object in the category of presheaves as follows: Let $\tilde{+} : h_M \times h_M \to h_M$ be the composition

$$h_M \times h_M \cong h_{M\times M} \overset{h(+)}{\to} h_M \tag{E.1.4}$$

where the first map is the natural isomorphism given by taking an element of $(h_M \times h_M)(-)$ say $f_1, f_2 : - \to M$ to the unique morphism $(f_1, f_2) : - \to M \times M$ such that $f_i = pr_i \circ (f_1, f_2)$ for $i = 1, 2$. Further let $\tilde{0} : h_* \to h_M$ be given by

$$\tilde{0} = h(0). \tag{E.1.5}$$

Then one checks straightforwardly that $\tilde{+}$ and $\tilde{0}$ satisfy the conditions of the addition and zero map respectively. Note that this implies that if T is any object of $\mathcal{C}$ then if $0_T \in h_M(T)$ is the image of $\tilde{0}(T)$ we have that the triple

$$(h_M(T), \tilde{+}(T), 0_T) \tag{E.1.6}$$

is a commutative monoid. Furthermore since the maps $\tilde{+}, \tilde{0}$ are natural transformations the presheaf of sets h_M can in fact be considered as a presheaf of commutative monoids in this way.

Furthermore if t is any Grothendieck topology on $\mathcal{C}$ and we let $\mathrm{L}_t(M)$ denote the sheafification of a representable functor h_M, then we claim that the sheaf $\mathrm{L}_t(M)$ of sets can in fact be considered as a sheaf of monoids in a canonical way. Indeed since the canonical map $h_M \times h_M \to \mathrm{L}_t(M) \times \mathrm{L}_t(M)$ is a sheafification of $h_M \times h_M$, then letting $\hat{+} : \mathrm{L}_t(M) \times \mathrm{L}_t(M) \to \mathrm{L}_t(M)$ be the map induced from the composition of $\tilde{+}$ with the canonical sheafification map $h_M \to \mathrm{L}_t(M)$ and setting $\hat{0} := \mathrm{L}_t(\tilde{0}) : \mathrm{L}_t(*) \to \mathrm{L}_t(M)$, one proves the claim almost immediately from the universal properties. In other words the sheaf $\mathrm{L}_t(M)$ together with $\hat{+}$ as addition and $\hat{0}$ as zero gives a monoid object in the category of sheaves in the topology t and $\mathrm{L}_t(M)$ can be considered a presheaf of commutative monoids. Note also that in this way the sheafification map $h_M \to \mathrm{L}_t(M)$ is a morphism of presheaves of monoids.

Extension of scalars

Let Λ be any commutative semiring and $\mathcal{C}$ be a category with products and terminal object. If M is a commutative monoid object in $\mathcal{C}$ we have seen that then h_M can be considered a presheaf of commutative monoids, and moreover since $- \otimes_{\mathbb{N}} \Lambda$ is a functor we get induced a presheaf of Λ-semimodules

$h_M \otimes_{\mathbb{N}} \Lambda : \mathcal{C}^{op} \to Semimod_\Lambda$. Similarly if t is any Grothendieck topology on $\mathcal{C}$ we get a presheaf $\mathrm{L}_t(M) \otimes_{\mathbb{N}} \Lambda$ on the category $\mathcal{C}$ of Λ-semimodules.

E.2 Constructing monoid objects in distributive categories

Let us recall the notion of a distributive category.

Definition E.2.1. A category $\mathcal{C}$ with finite products and coproducts is called *(finitary) distributive* if for any $X, Y, Z \in \mathcal{C}$ the canonical distributivity morphism

$$X \times Y \coprod X \times Z \to X \times (Y \coprod Z) \tag{E.2.1}$$

is an isomorphism. The canonical morphism is the unique morphism such that $X \times Y \to X \times (Y \coprod Z)$ is $X \times i$, where $i : Y \to Y \coprod Z$ is the coproduct injection, and similarly for $X \times Z \to X \times (Y \coprod Z)$.

A category $\mathcal{C}$ with finite products and all small coproducts is *infinitary distributive* if the statement applies to all small coproducts.

The following Lemma tells us that the category of S-schemes is an example of an infinitary distributive category.

Lemma E.2.2. *Suppose $i \in I$ is an index set and let $q_i : X_i \to S$ be a morphism for each $i \in I$. Further let $X \to S$ be any morphism of schemes. Then the pair*

$$((p_{i,1})_{i \in I} : \coprod_{i \in I}(X \times_S X_i) \to X), \coprod_{i \in I} p_{i,2} : \coprod_{i \in I}(X \times_S X_i) \to \coprod_{i \in I} X_i) \tag{E.2.2}$$

where for each $i \in I$ the maps $p_{i,1} : X \times_S X_i \to X$, $p_{i,2} : X \times_S X_i \to X_i$ is the first and second projection respectively , is a pullback of the diagram

$$\begin{array}{ccc} \coprod_{i \in I} X \times_S X_i & \xrightarrow{\coprod_{i \in I} p_{i,2}} & \coprod_{i \in I} X_i \\ \downarrow{\scriptstyle (p_{i,1})_{i \in I}} & & \downarrow{\scriptstyle (q_i)_{i \in I}} \\ X & \longrightarrow & S. \end{array} \tag{E.2.3}$$

Proof. Suppose we are given morphisms of S-schemes $f : T \to \coprod X_i$ and $g : T \to X$. Then for each $i \in I$ we get induced a morphism $f_i : f^{-1}(X_i) \to X \times_S X_i$ such that $pr_X \circ f_i = g|_{f^{-1}(X_i)}$ and $pr_{X_i} \circ f_i = f|_{f^{-1}(X_i)}$ and so we canonically get morphisms $f_i' : f^{-1}(X_i) \to \coprod_{i \in I} X \times_S X_i$. Since f_i', f_j' coincide on $f^{-1}(X_i) \cap f^{-1}(X_j)$ the f_i' glue to give a unique morphism $f' : T \to \coprod_{i \in I} X \times_S X_i$ and it is clear from the construction that we have $\coprod_{i \in I} p_{i,2} \circ f' = f$ and $g = (p_{i,1})_{i \in I} \circ f'$. $\square$

The Category of Sets, Presheaves (on a given category $\mathcal{C}$) and sheaves in a Grothendieck topology on a category $\mathcal{C}$, are other examples of infinitary distributive categories that appear in this thesis.

Construction E.2.3. Let $\mathcal{C}$ be an infinitary distributive category with finite products and a terminal object $* \in \mathcal{C}$. Suppose that for each $n \in \mathbb{N}$ we have an object $M_n \in \mathcal{C}$ with $M_0 = *$ and that for each pair of indices $m, n \in \mathbb{N}$ we have maps $\alpha^{m,n} : M_m \times M_n \to M_{m+n}$ such that for all non-negative integers m, n, k the following diagrams commute:

1. :

$$\begin{tikzcd} M_m \times M_n \arrow[r, "\alpha^{m,n}"] \arrow[d, "swap"] & M_{m+n} \arrow[d, equal] \\ M_n \times M_m \arrow[r, "\alpha^{n,m}"] & M_{m+n}. \end{tikzcd} \tag{E.2.4}$$

2. :

$$\begin{tikzcd} M_k \times M_m \times M_n \arrow[r, "M_k \times \alpha^{m,n}"] \arrow[d, "\alpha^{k,m} \times M_n"] & M_k \times M_{m+n} \arrow[d, "\alpha^{k,m+n}"] \\ M_{k+m} \times M_n \arrow[r, "\alpha^{k+m,n}"] & M_{m+n+k}. \end{tikzcd} \tag{E.2.5}$$

3.

$$\begin{tikzcd} M_n \arrow[r, "\cong"'] \arrow[rr, bend left, equal] & M_0 \times M_n \arrow[r, "\alpha^{0,n}"'] & M_n. \end{tikzcd} \tag{E.2.6}$$

Set $M := \coprod_{n \in \mathbb{N}} M_n$ and let $j_n : M_n \to M$ be the canonical map. Set $0 := j_0 : M_0 = * \to M$ and note that since the category is infinitary distributive we have $M \times M \cong \coprod_{m,n \in \mathbb{N}} M_m \times M_n$ hence the maps $\alpha^{m,n}$ induce a unique map $+ : M \times M \to M$ making the following diagram commutative:

$$\begin{tikzcd} M \times M \arrow[r, "+"] & M \\ M_m \times M_n \arrow[r, "\alpha^{m,n}"] \arrow[u, "j_m \times j_n"] & M_{m+n}. \arrow[u, "j_{m+n}"'] \end{tikzcd} \tag{E.2.7}$$

Then it is not hard to show that M together with the maps 0 and + form a commutative monoid object in $\mathcal{C}$. Indeed commutativity and associativity of addition follows almost immediately from Item 1 and Item 2 respectively. To check commutativity of the diagram given in Equation (E.1.3) is satisfied note that the composition $M_n \to * \times M \overset{0 \times M}{\to} M \times M$ factors as $M_n \to * \times M_n \overset{j_0 \times j_n}{\to} M \times M$ and similarly $M_n \to M \times * \overset{M \times 0}{\to} M \times M$ factors through $j_n \times j_0$. From Item 1 and Item 3 the claim now follows.

Definition E.2.4. Let $\mathcal{C}$ be an infinitary distributive category with finite products and a terminal object $* \in \mathcal{C}$. Then a *graded commutative monoid object* in $\mathcal{C}$ is any object constructed by means of Construction E.2.3.

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Nomenclature

$(L_1 \cdots L_m \cdot F)$	intersection number, page 164
$(X, G)/S$	pair satisfying the conditions of Lemma 3.3.1, page 116
$(X/S)^d$	self fibered product over a base, page 34
$(x_0, x_1)^*(\mathcal{Z})$	cycle associated with a cycle and a fat point, page 53
$(x_0, x_1)^*(Z/S)$	cycle associated with a fat point and a closed subscheme, page 53
(x_0, x_1, R)	fat point, page 49
$[K : k]_{\text{insep}}$	inseparable degree, page 15
$[K : k]_{\text{sep}}$	separable degree, page 15
$\text{Aut}_S(X)$	group of S-automorphisms , page 20
$\mathcal{A}^G$	subsheaf of G-invariants of the sheaf of algebras $\mathcal{A}$, page 28
$\mathcal{F}^s_\tau$	separation of the presheaf $\mathcal{F}$, page 239
$\chi(X, \mathcal{F})$	Euler characteristic, page 164
$\text{CH}(X, r)$	Chow group of algebraic cycles, page 90
$C^{irr}{}_{r,d}((X, i)/S)$	geometrically irreducible hypersurfaces contained in the Chow scheme, page 186
$C_{r,d}((X, i)/S)$	Chow scheme, page 189
$C_{r,d}((X, i)/S)$	underlying set of the Chow scheme, page 185
$C_r((X, i)/S)$	Chow monoid, page 195
$\text{cycl}(f)$	pullback of relative cycles along f, page 69
$\text{Cycl}(X)$	group of cycles on a scheme, page 38

$\mathrm{Cycl}(X/S,r)$	group of relative cycles, page 54
$\mathrm{Cycl}(X/S,r)_{\mathbb{Q}}$	presheaf of relative cycles, page 75
$\mathrm{Cycl}(X/S,r)_{UI}$	presheaf of relative cycles with universally integral coefficients, page 76
$\mathrm{Cycl}^{eff}(X/S,r)$	monoid of effective relative cycles, page 54
$\mathrm{Cycl}^{eff}(X/S,r)_{\mathbb{Q}_+}$	presheaf of effective relative cycles, page 75
$\mathrm{Cycl}^{eff}_D(\mathbb{P}^I_S/S,r)_{UI}$	relative cycles of multi-degree D, page 168
cycl_X	homomorphism from closed subschemes to cycles on a scheme, page 39
$\mathrm{Cycl}_{equi}(X/S,r)$	group of equidimensional relative cycles, page 54
$\mathrm{Cycl}_{equi}(X/S,r)_{UI}$	presheaf of equidimensional relative cycles with universally integral coefficients, page 76
$\mathrm{Cycl}^{\mathit{eff}}_d((X,i)/S,r)_{UI}$	relative cycles of degree d with respect to a closed embedding, page 182
$\mathrm{Cycl}^{\mathit{eff}}(X)$	monoid of effective cycles on a scheme, page 39
$\mathrm{Cycl}_{\mathrm{equi}}(X/S,r)_{\mathbb{Q}}$	presheaf of equidimensional relative cycles, page 75
$\mathcal{Z} \overset{ess}{=} \mathcal{W}$	essentially the same cycles, page 92
$\deg_s(\mathcal{Z})$	multi-degree of $\mathcal{Z}_s$, page 167
$\delta_{X/S}$	diagonal morphism, page 19
$\Delta_{X/S}$	diagonal morphism, page 19
$\dim(X/S)$	local fiber dimension function, page 2
$\dim_x(X)$	dimension of a topological space at a point, page 1
$\mathrm{Eff}_S(\mathcal{Z})$	effective locus of a relative cycle, page 96
η	fixed natural transformation in Chapter 4, page 122
$\mathrm{Exp.\,Char}(S)$	exponential characteristics of a scheme, page 113
$\mathrm{exp.\,char}(k(s))$	exponential characteristic of a field, page 113
$\mathrm{c}_1(L)$	first Chern class in terms of K_0, page 164
$\iota_k(b)$	formal tensor conjugate, page 33

$\mathrm{K}(X)$	Grothendieck group of X, page 164
$\mathrm{Hilb}(X/S,r)$	closed subschemes flat and equidimensional over the base, page 60
${\mathrm{H}^{irr}}_{\underline{d},\underline{n}}$	presheaf of geometrically irreducible equidegree hypersurfaces, page 177
$\mathrm{H}_{\underline{d},\underline{n}}$	presheaf of equidegree hypersurfaces, page 177
$\mathbb{H}_{r,n}$	graded commutative monoid via equidegree hypersurfaces, page 177
$\mathrm{length}_A(M)$	length of the A-module M, page 213
$\lim_{f\to s}(Z/S)$	cycle-theoretic fiber, page 91
$\mathrm{M}_\eta(A)$	η-construction for a ring, page 123
$\mathbb{N}(\mathrm{Hilb}(X/S,r)$	free monoid of closed subschemes flat and equidimensional over the base, page 60
$\mathbb{N}(\mathrm{PropHilb}(X/S,r))$	free monoid of closed subschemes flat, proper and equidimensional over the base , page 60
$\mathbb{N}_t(X/S)$	sheafification of $N(X/S)$, page 201
$\mathrm{N}_\eta(A)$	product of residue fields with respect to F, page 122
$\mathrm{Null}_S(\mathcal{Z})$	vanishing locus of a relative cycle, page 97
ν_p	p-adic valuation on $\mathbb{Q}$, page 159
$(-)^{Perf}$	natural transformation from identity to perfect closure, page 18
k^{Perf}	perfect closure, page 18
$\pi_*\mathcal{O}_X^G$	subsheaf of G-invariants, page 23
$\mathbb{P}_S^I$	multi projective space, page 165
$\mathrm{PropCycl}(X/S,r)$	group of proper relative cycles, page 54
$\mathrm{PropCycl}(X/S,r)_{\mathbb{Q}}$	presheaf of proper relative cycles, page 75
$\mathrm{PropCycl}(X/S,r)_{UI}$	presheaf of proper relative cycles with universally integral coefficients, page 76
$\mathrm{PropCycl}^{eff}(X/S,r)$	monoid of effective proper relative cycles, page 54

$\mathrm{PropCycl}^{eff}(X/S,r)_{\mathbb{Q}_+}$	presheaf of effective proper relative cycles, page 75
$\mathrm{PropCycl}_{equi}(X/S,r)$	group of proper equidimensional relative cycles, page 54
$\mathrm{PropCycl}_{equi}(X/S,r)_{UI}$	presheaf of proper equidimensional relative cycles with universally integral coefficients, page 76
$\mathrm{PropCycl}_{\mathrm{equi}}(X/S,r)_{\mathbb{Q}}$	presheaf of proper equidimensional relative cycles, page 75
$\mathrm{PropHilb}(X/S,r)$	closed subschemes flat, proper and equidimensional over the base, page 60
$\mathrm{Psh}(\mathcal{C})$	category of presheaves on $\mathcal{C}$, page 238
$\psi_{X/S}$	graph of the pair $(X,G)/S$, page 21
$(K)_{pi}$	purely inseparable closure, page 15
$\mathrm{Rat}(X,r)$	subgroup of cycles rationally equivalent to zero, page 90
$i_{B/A}$	projection from Section 4.1, page 129
$(B/A)_\eta$	relative η-construction for rings, page 129
$(Z/X)^\eta$	relative η-construction for schemes, page 141
$\mathscr{M}_\eta(Z/X)$	defining sheaf for the scheme theoretic relative η-construction, page 141
$q_{B/A}$	induced map $A \to (B/A)_\eta$, page 129
$u_{B/A}$	projection from Section 4.1, page 129
$\mathbb{H}^{irr}{}_{\underline{d},\underline{n}}$	scheme representing $\mathrm{H}^{irr}{}_{\underline{d},\underline{n}}$, page 177
$\mathbb{H}_{\underline{d},\underline{n}}$	scheme representing $\mathrm{H}_{\underline{d},\underline{n}}$, page 177
$\mathrm{L}_{prop}(X/S)$	sheafification of representable in the proper topology, page 191
$\mathrm{Sh}(\mathcal{C})$	category of sheaves on $\mathcal{C}$, page 238
L_t	sheafification of a representable presheaf, page 153
$\mathcal{F}_{prop}$	sheafification in the proper topology, page 191
Σ_d	group of permutations, page 32
$\mu^\eta X$	map associated to the scheme theoretic η-construction, page 137

$\mathscr{N}_\eta(X)$	sheaf of products of residue fields with respect to F, page 135
$\mathrm{Stab}_G(a)$	stabilizer of the element a, page 23
$\rho_k(b)$	elementary symmetric tensor, page 33
q_X	algebra structure on $\mathscr{M}_\eta(X)$, page 136
t_X	algebra structure on $\mathscr{N}_\eta(X)$, page 135
$\mathrm{supp}(\mathscr{Z})$	support of a cycle, page 39
$\mathrm{Sym}^d(X/S)$	d'th symmetric power over the base S, page 35
$\mathrm{S}_n(M/A)$	symmetric tensors, page 32
$\mathrm{Sym}^\bullet(X/S)$	monoid of symmetric powers of a scheme over a base, page 38
$(M/A)^{\otimes n}$	n-fold tensor product, page 32
q_A	natural map from Section 4.1, page 123
t_A	natural map from Section 4.1, page 122
$\underline{g}$	induced morphism to the scheme-theoretic relative η-construction, page 141
$\mathscr{X}_d$	universal degree d hypersurface in $((\mathbb{P}^n)^\vee)^{r+1}$, page 184
$\upsilon_{Y,X/S}$	morphism of Corollary 3.3.6, page 118
$\mathbb{Z}(\mathrm{Hilb}(X/S,r))$	free group of closed subschemes flat and equidimensional over the base, page 60
$\mathbb{Z}(\mathrm{PropHilb}(X/S,r))$	free group of closed subschemes flat, proper and equidimensional over the base, page 60
$Z(H)$	attempted Chow inverse, page 183
AF	AF-property, page 30
B^G	ring of G-invariants, page 22
$Ch(Z)$	integral subscheme $p(f^{-1}(Z)) \subset G/S.$, page 181
$Chow(i)$	Chow homomorphism restricted to relative cycles on a closed subscheme, page 183
$Chow_d(i)$	degree d Chow homomorphism restricted to relative cycles on a closed subscheme, page 183

F	fixed endofunctor in Chapter 4, page 122
$f_*(\mathcal{Z})$	push-forward of the cycle $\mathcal{Z}$ along f, page 82
G_X	object associated to the pair (X, G), page 20
$h_{\mathcal{U}}$	sieve associated with the covering $\mathcal{U}$, page 242
K_{sep}	separable closure, page 15
$N(X/S)$	presheaf of abelian monoids freely generated by h_X, page 201
$p^*(\mathcal{Z})$	flat pullback of a cycle, page 39
$p_*(\mathcal{Z})$	proper pushforward of a cycle, page 42
$Q(A)$	total ring of fractions, page 11
R^h	Henselization of the local ring R, page 227
R^{sh}	strict Henselization of the local ring R, page 227
V_η	fixed subcategory of valuation rings from Section 4.1, page 123
X/G	quotient by a group, page 21
X_Z	blowup of X with center in Z, page 102
Chow	homomorphism taking relative cycles of dimension r in projective space to codimension one relative cycles in multi-projective space, page 180

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